Introductory fluid mechanics

Torenbeek & Wittenberg Chs 2 & 3
Anderson Ch 2

Dimensions and units in fluid mechanics

A **dimension** is a qualitative aspect of a physical variable (e.g. mass, length, time, force, pressure).

A system of **units** is a way of assigning numerical values to a given quantity of a physical variable.

The quality and quantity of a physical entity (i.e. its dimension and amount) stays the same as we change the system of units, but the numerical value that gets assigned can change.

E.g. 10m or 32.81ft describes the same quantity of length using two different unit systems.

We only need a small number of primary quantities to describe problems in mechanics.
We work with mass, length, time and temperature (other choices are possible) and use SI units:

Quantity	Dimension	SI Unit
Mass	[M]	kg
Length	[L]	m
Time	[T] (**Not** *symbol* for temperature variable, *T*.)	s
Temperature	[θ]	K

Note the use of the [ ] operator to signify the dimensions of a quantity.

Quantities may be dimensionless (e.g. angles), expressed as [1].

The dimensions of derived quantities such as area, pressure can all be expressed in terms of the primary quantities raised to some power. Sometimes names are given to the *units* of the derived quantities (e.g. Pascal for pressure) – just as a kind of shorthand – this has no effect on the dimensions.

[pressure] = [force/area] = [mass×acceleration/area] = $[MLT^{-2}/L^2] = [M/LT^2]$

Dimensions of variables and equations in fluid mechanics

Examples of derived quantities with dimensions and SI units.

Quantity	Dimension	SI Unit
Area	$[L^2]$	m^2
Volume	$[L^3]$	m^3
Velocity	$[LT^{-1}]$	m/s
Acceleration	$[LT^{-2}]$	m/s^2
Density ρ	$[ML^{-3}]$	kg/m^3
Force	$[MLT^{-2}]$	$N = kg\,m/s^2$
Pressure, stress	$[ML^{-1}T^{-2}]$	$Pa = N/m^2 = kg/(m.s^2)$
Frequency	$[T^{-1}]$	$Hz = 1/s$
Angle	$[LL^{-1}] = [1]$	rad (dimensionless)
Angular velocity	$[T^{-1}]$	rad/s
Energy, work	$[ML^2T^{-2}]$	$J = N.m$
Power	$[M^2T^{-3}]$	$W = J/s$
Specific heat, gas constant	$[L^2T^{-2}\theta^{-1}]$	$m^2/(s^2.K)$
Viscosity μ	$[ML^{-1}T^{-1}]$	$Pa.s = kg/(m.s)$
Kinematic viscosity $\nu = \mu/\rho$	$[L^2T^{-1}]$	m^2/s
Diffusivity	$[L^2T^{-1}]$	m^2/s

An apparently simple but far-reaching truth is that equations in mechanics have to have dimensional as well as numerical equality.

For example, take $F=ma$. Obviously the numerical values should agree, but also the dimensions will. $[F]=[ma]$, i.e. $[MLT^{-2}] = [M \times LT^{-2}]$. ✔ This property is called dimensional homogeneity.

A very useful check if you rearrange any equation in mechanics is to ensure the equation still has dimensional homogeneity. If it doesn't, you made a mistake.

You should always state the units as well as numerical value of any calculation. E.g. F = 134 N.

Unit conversion factors

We will typically work in SI units but always in this area one has to convert between British Engineering System and SI units, owing to the fact that international aviation agreements adopted measurement of distance in nautical miles, altitude in feet. Also, many texts use non-SI unit systems.

Quantity	British Engineering System unit	SI unit	Conversion factor (Multiply British Engineering system to get SI value)
Length	ft	m	0.3048
Mile	5280 ft	1.609 km	—
Nautical mile (NM)	6080 ft	1.853 km	—
Area	ft^2	m^2	0.09290
Mass	lbm slug	kg kg	0.4536 14.59
Force	lbf	N	4.448
Pressure and Stress	lbf/ft^2 (psf) $lbf/in.^2$ (psi)	N/m^2 (Pa) kN/m^2 (kPa)	47.88 6.895
Density	lbm/ft^3	kg/m^3	16.02
Temperature difference	°R	K	1/1.8
Specific enthalpy and fuel heating value	Btu/lbm	kJ/kg	2.326
Specific heat (c_p, c_v)	Btu/(lbm-°R)	kJ/(kg-K)	4.187
Gas constant ($g_c R$)	$ft^2/(s^2$-°R)	$m^2/(s^2$-K)	0.1672
Rotational speed	rpm	rad/s	$2\pi/60 = 0.1047$
Specific thrust ($F/\dot{m}$)	lbf/(lbm/s)	N-s/kg = m/s	9.807
Thrust specific fuel consumption (S)	$\frac{\text{lbm fuel/h}}{\text{lbf thrust}} = \frac{\text{lbm}}{\text{lbf-h}}$	$\frac{\text{mg fuel/s}}{\text{N thrust}} = \frac{\text{mg}}{\text{N-s}}$	28.33
Power	hp Btu/hr	W W	745.7 0.2931
Power specific fuel consumption (S_P)	$\frac{\text{lbm fuel/h}}{\text{hp}}$	$\frac{\text{mg/s}}{\text{W}} = \frac{\text{mg}}{\text{W-s}}$	0.1690

$1\,\text{kt} = 1\,\text{nm/hr} \approx 100\,\text{ft/min}$
$1\,\text{kt} = 0.5148\,\text{m/s} = 1.853\,\text{km/hr}$

$1\,\text{US gal} = 3.785\,\text{l}$
$1\,\text{UK gal} = 4.546\,\text{l}$

$T(\text{K}) = T(°\text{C}) + 273.16$

Unit conversion methodology

To convert the numeric value of a physical quantity from one system of units to another is a simple matter once we know the conversion factors for all the quantities involved.

E.g. convert a pressure of 32 p.s.i. (lbf/in²) to Pa (N/m²).

$$32\,\text{psi} = 32\,\frac{\text{lbf}}{\text{in}^2}$$

$$= 32\,\frac{\cancel{\text{lbf}}}{\cancel{\text{in}^2}} \times 4.448\,\frac{\text{N}}{\cancel{\text{lbf}}} \times \frac{1}{(25.4\times10^{-3})^2}\,\frac{\cancel{\text{in}^2}}{\text{m}^2}$$

$$= 32 \times \frac{4.448}{(25.4\times10^{-3})^2}\,\frac{\text{N}}{\text{m}^2}$$

$$= 32 \times 6.894 \times 10^3\,\frac{\text{N}}{\text{m}^2}$$

$$= 32 \times 6.894\,\frac{\text{kN}}{\text{m}^2}$$

(cf. conversion factor of 6.895 in last table.)

$$= 220.6\,\text{kPa}$$

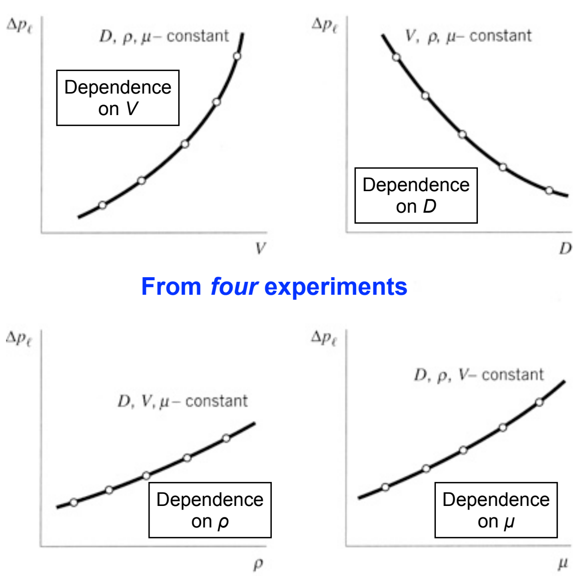
Dimensionless variables

Arising from the fact that equations in mechanics have to have dimensional as well as numerical equality we find that we can typically reduce the number of independent variables and so number of experiments that are needed to understand a functional relationship.

E.g. supposing we wanted to find the dependence of pressure drop per unit length Δp_l for flow through a pipe, diameter *D,* at any average speed *V* for fluid of density ρ and viscosity μ. We might guess that $\Delta p_l = f(D, \rho, \mu, V)$

Using the fact that this relationship must be dimensionally homogeneous we can establish that instead of a functional relationship between five dimensional variables, there is an equivalent relationship between only two dimensionless variables

$$\frac{D\Delta p_l}{\rho V^2} = \phi\left(\frac{\rho V D}{\mu}\right)$$

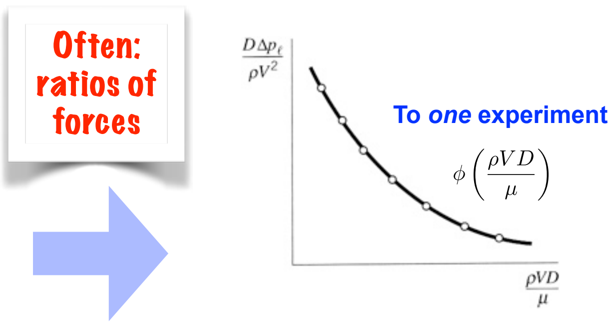


This greatly reduces the number of experiments we need to understand/describe the relationship.

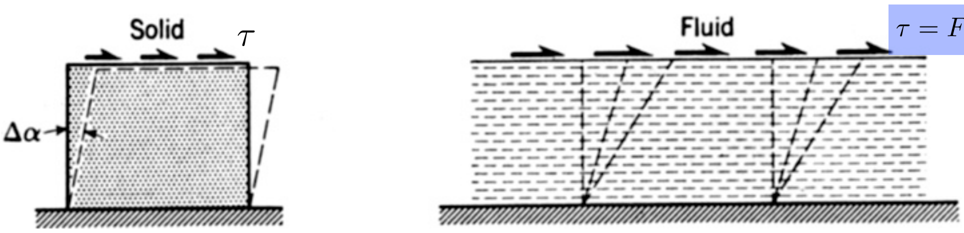
Partly for this reason dimensionless variables are very often used in fluid mechanics.

We will see a few of the more common ones.

E.g. $\rho VD/\mu$ is called the Reynolds number.



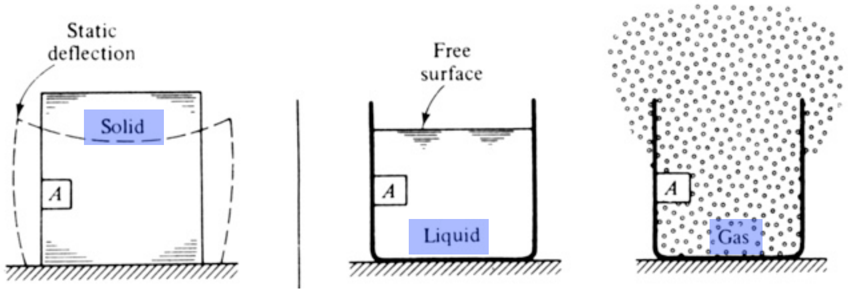
Fluids vs solids



$\tau = F_{\text{tangential}}/\text{Area}$

Shames

1. **Definition: Solids** *at rest* can support a shear stress (tangential force per unit area), while **fluids** cannot. Shear stress is typically given the symbol τ (tau).
2. Fluids subjected to a shear stress deform continually until (if ever) the shear stress reaches zero everywhere. However, even at rest, they can support normal stress (pressure).
3. Both liquids and gases are fluids. In aerospace applications we most often consider gases (especially air). Gases are much more compressible than liquids.
4. **Definition:** Liquids will form a distinct surface and occupy a fixed volume at given pressure, while gases expand to fill a volume.



White

Fundamental variables in fluid mechanics

The four fundamental variables of aerospace fluid mechanics according to Anderson are

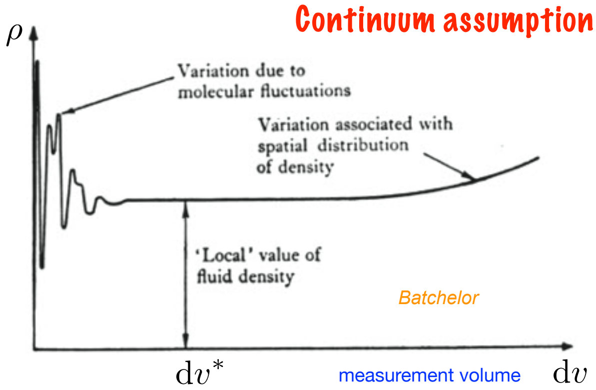
1. Density ρ
2. Pressure p
3. Temperature T
4. Flow velocity $\boldsymbol{V}$

To which we could add a fifth, a fluid property that is principally a function of temperature

5. Viscosity μ Viscosity provides the linkage between fluid motion and shear stress: friction.

In reality fluids are composed of molecules in motion, but we make the **continuum assumption** that length scales of interest are large enough that we can take average properties over a very large number of molecules.

Continuum assumption



Consider characterising some quantity as a function of position in space (e.g. the density, ρ) at some instant in time

Need to locally average to eliminate `statistical' fluctuations produced by random (e.g. Brownian) molecular motion

As the averaging volume d*v* increases, a value at which statistical/molecular fluctuation is imperceptible is reached: d*v**; at this volume we have the **local density**

Typically d*v** is very small, say of order 10^{-9}mm^3

At larger volumes, variations may start to arise owing to spatial variation in the average value

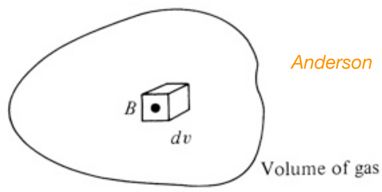
We then can use the ideas of calculus to deal with gradients of these averaged local properties.

Density and pressure

Density of a substance (including a fluid, or gas) is the mass of that substance per unit volume.

If *m* is mass and *v* is volume, then $\rho = \lim_{\mathrm{d}v \to 0} \frac{\mathrm{d}m}{\mathrm{d}v}$

i.e. we conceptually obtain the density via the continuum assumption – but it is simple enough to measure the mass of a finite volume of gas or liquid.



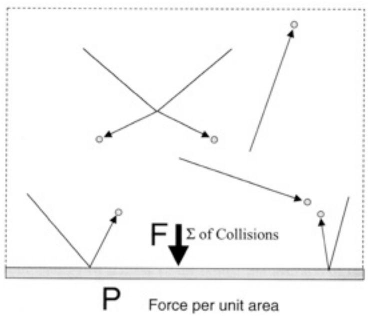
Anderson

Density is a function of location in space and may vary in time, so ρ=ρ(*x*,*y*,*z*,*t*) in general.

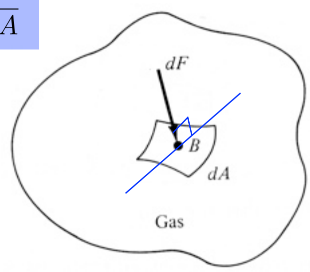
Dimensions of density are those of mass per unit volume, in SI units this is kg/m³. The standard density of air at sea level is 1.225 kg/m³. For comparison the density of pure water is 998kg/m³.

Pressure is the *normal* (i.e. perpendicular) force per unit area (traction or stress) exerted on a surface owing to the time rate of change of momentum of gas (or fluid) molecules impacting on the surface.

Brandt et al.



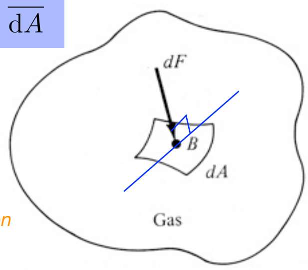
$$p = \lim_{\mathrm{d}A \to 0} \frac{\mathrm{d}F}{\mathrm{d}A}$$



Anderson

Pressure properties

$$p = \lim_{\mathrm{d}A \to 0} \frac{\mathrm{d}F}{\mathrm{d}A}$$



Anderson

Pressure is by definition a scalar, isotropic quantity.

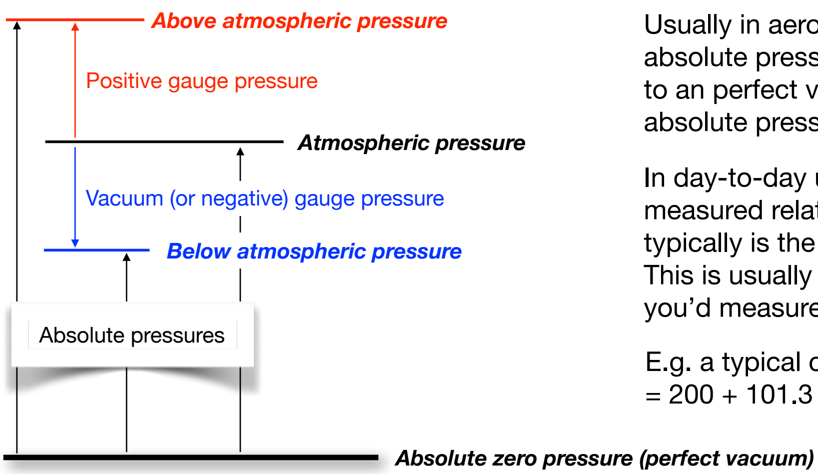
In concept there is no difference whether the surface on which pressure is exerted is solid or not, so it could be drawn inside the fluid.

Pressure also exists within the surrounding fluid and in general varies from point to point in space and in time, i.e $p = p(x, y, z, t)$.

Pressure (as distinct from shear stress) can exist in a fluid at rest.

Dimensions of pressure are those of force per unit area: in SI units this is N/m² or Pa. Another common unit is atm, 1atm = 101,325Pa.

Note the convention: positive pressure is a compressive stress.

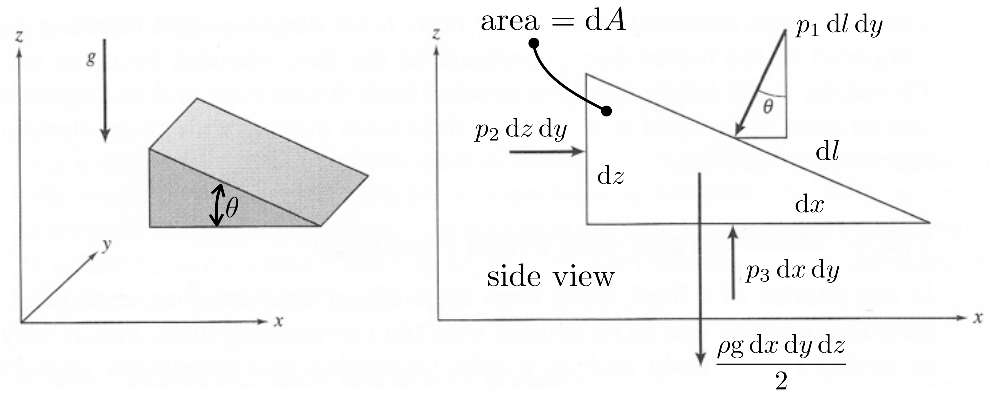


Usually in aerospace engineering we work with absolute pressure, or pressure measured relative to an perfect vacuum. (Note that negative absolute pressures cannot be obtained in fluids.)

In day-to-day usage we typically consider pressure measured relative to a background value, which typically is the surrounding atmospheric pressure. This is usually called a 'gauge pressure' – what you'd measure with a differential pressure gauge.

E.g. a typical car tyre pressure is 200kPa (gauge) = 200 + 101.3 kPa (absolute) = 301.3kPa.

Pressure is isotropic and scalar



Smits

Consider pressure and gravitational forces acting on an <u>infinitesimal</u> wedge of fluid at rest.

First, suppose that p_1, p_2, p_3 might be different – that pressure depends on direction.

Horizontal equilibrium

$(p_1 \mathrm{d}l \sin\theta = p_2 \mathrm{d}z)\mathrm{d}y$ and $\mathrm{d}z = \mathrm{d}l \sin\theta$ | Conclude $p_1 = p_2$

Vertical equilibrium

$(p_1 \mathrm{d}l \cos\theta + \rho g \mathrm{d}A = p_3 \mathrm{d}z)\mathrm{d}y$ and $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}z/2$ and $\mathrm{d}x = \mathrm{d}l \cos\theta$

or $\mathrm{d}y\,\mathrm{d}l \cos\theta(p_1 + \rho g\,\mathrm{d}z/2 = p_3)$ so in $\lim \mathrm{d}z \to 0$ | Conclude $p_1 = p_3$

Hence $p_1 = p_2 = p_3 = p$, independent of direction. I.e. it's *isotropic*.

Also p must be a scalar quantity (only 1 value needed). But $p(x, y, z, t)$ may vary in space and time!

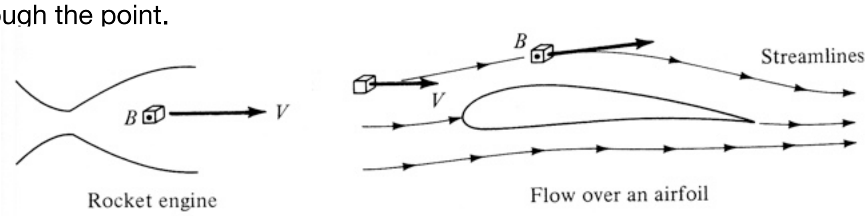
Temperature and velocity

Temperature is a measure of the average kinetic energy of the particles (molecules) in the gas as they move around and collide with one another.

Units of temperature in SI are Kelvin (K), which is an absolute temperature scale. Degrees Celcius °C are also commonly used, with the zero reference being the only difference. 0K= -273.16°C. The standard atmosphere sea-level temperature is 288.16K (15°C).

The temperature can also vary in space (e.g. with altitude) and in time.

The **velocity** at any fixed point in a fluid is the velocity of an infinitesimally small volume of fluid as it sweeps through the point.



Anderson

The velocity $\boldsymbol{V}(x,y,z,t)$ <u>is a vector</u>, so has three components: $\boldsymbol{V}=(u,v,w)(x,y,z,t)$ but often (and a little incorrectly) we state a scalar value V called velocity when we really mean the *speed* or *velocity magnitude*, i.e. $V = |\boldsymbol{V}|$.

Dimensions of velocity are those of length divided by time. In SI units this is m/s.

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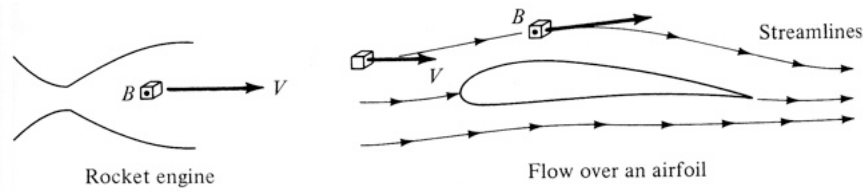
FLUID MECHANICS

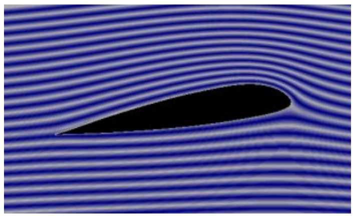
Kinematics – flow studies related to velocity field



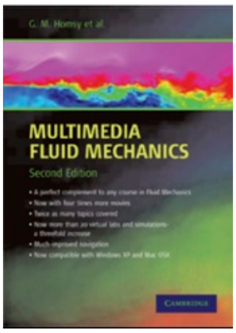
Anderson

By connecting up the local velocity vectors with lines that are locally tangential to the vector field we draw trajectories that are called *streamlines*. If the flow is steady in time, these streamlines are the same as the paths that particles (e.g. dye particles) take as they move through the fluid.

These *pathlines* are often used in experiments as indicators of streamlines.

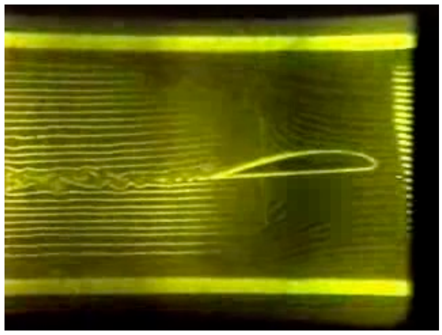


(Note that if the flow is unsteady in time, pathlines are not the same as streamlines, but in this course we only deal with flows that are steady – at least, on average – in time.)



Flow visualization provides a great way to rapidly comprehend flows which would otherwise require a complicated formal description.

View the 'Kinematics' section of the MFM DVD for more information.



http://www.youtube.com/watch?v=ouF9Xkoi3uk&NR=1

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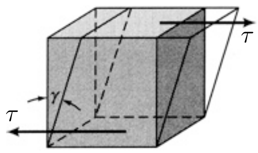
FLUID MECHANICS

Viscosity and viscous stress

The **viscosity** of a fluid supplies the relationship between the spatial *gradient* of velocity and tangential traction or shear stress.

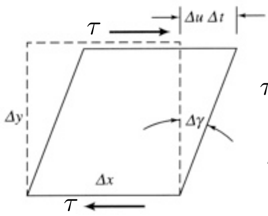
For a solid material under shear deformation, the shear stress (τ) is proportional to shear strain ϒ.



$\tau \propto \gamma$ or $\tau = G\gamma$

G is called the solid's *shear modulus*.

For a fluid under shear deformation, shear stress is proportional to shear strain rate.



$\tau \propto \dfrac{\mathrm{d}\gamma}{\mathrm{d}t}$ or $\tau = \mu \dfrac{\mathrm{d}\gamma}{\mathrm{d}t}$

***μ* is called the fluid's viscosity.**

Since $\Delta\gamma\Delta y = \Delta u \Delta t$ where *u* is a velocity component

in the limit as Δy→0 we have $\dfrac{\mathrm{d}\gamma}{\mathrm{d}t} = \dfrac{\mathrm{d}u}{\mathrm{d}y}$ and then $\tau = \mu \dfrac{\mathrm{d}u}{\mathrm{d}y}$

Dimensions of viscosity are those of stress multiplied by time. Its SI units are Pa.s or kg/(m.s).

High → Low viscosity

Simply changing the viscosity can have a dramatic effect on the nature of a flow.

In this video we see four jet-type flows with exactly the same momentum/ buoyancy fluxes but with different viscosities in each case.

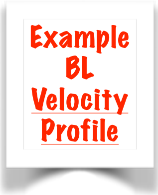
Viscosity of air at sea level is about 18×10^{-6} Pa.s (i.e. very small).

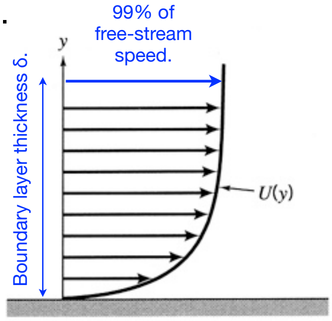
Effect of viscosity in wall *boundary layers*

Where it meets a solid boundary, fluid flow velocity matches that of the wall (often taken as zero).

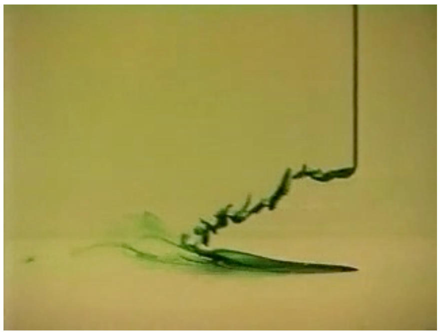
Moving out from the wall, local flow speeds change from the wall speed to the *free stream*, or far-field speed, through a region called the *boundary layer* (BL) where viscous effects are relatively strong.

The plot of flow speed vs distance from the wall is called a *velocity profile*.





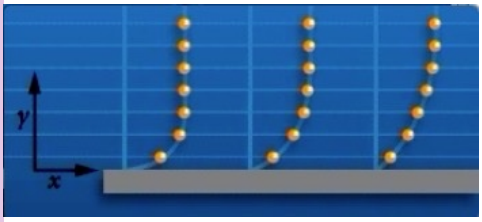
Boundary layer velocity profile near a solid wall.



http://www.youtube.com/watch?v=cUTkqZeiMow

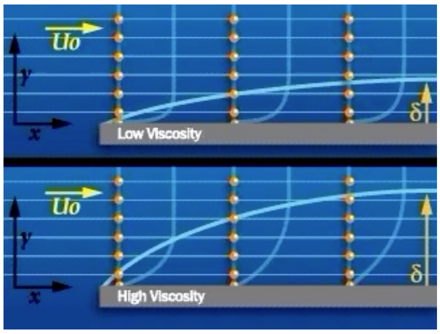
The thickness of the BL (δ) is often taken to be the distance from the wall at which 99% of the free-stream speed is reached.

Boundary layers in aeronautics are typically thin (say a few cm).



Flat plate BLs grow in depth with distance downstream.

Increasing viscosity increases growth rate.



Viscous shear stresses are highest at the wall and trend to zero outside the BL.

Equation of state for a perfect gas

Thermodynamics tells us that for any substance, only two of the three quantities *p, ρ, T* are independent.

The implication is that e.g. *p* = function (*ρ, T*), which is an *equation of state*. The two independent variables (arbitrary) are called the *state variables.*

For perfect gases (and to a very good approximation air near near the surface of the Earth is one) the relationship is especially simple:

$p = \rho \mathrm{R}T$ or $\rho = \dfrac{p}{\mathrm{R}T}$ where R is a constant specific to the gas in question.

For air, R=287.05 J/(kg K).

Example: what is air density if the local pressure is 75 kPa and the local temperature is –5 °C?

–5 °C = 273.16 – 5 K = 268.16 K.

ρ = 75×10^3 / (287.05 × 268.16) kg/m^3 = 0.9743 kg/m^3.

You may be more familiar with the universal gas constant *R* = 8314 J/(kmol K), the same for all gases – but in that case one also needs to know the molecular weight of the gas, which for air is M=28.96 kg/(kmol). One kmol contains 6.02×10^{26} molecules of gas. For any gas, R = R / M.

$$\mathrm{R}_{\mathrm{air}} = \frac{\mathcal{R}}{\mathcal{M}} = \frac{8314\,\mathrm{J/kmol\,K}}{28.96\,\mathrm{kg/kmol}} = 287.1\,\mathrm{J/kg\,K}$$

In thermodynamics, one often sees the symbol *v* for specific volume, or the volume occupied by one unit of mass. Note that the density (mass per unit volume) is the inverse of this, i.e. *ρ*=1/*v,* and so *pv*=R*T.*

FLUID MECHANICS

Speed of sound

The speed of sound in a medium is the speed at which an infinitesimal pressure pulse propagates in it.

The speed of sound in a gas, symbol *a*, is related to the average molecular speed as the molecules move around at random, *v*.

In turn this speed is related to the temperature of the gas, since the temperature is proportional to the average kinetic energy per unit volume of the gas.

It follows that $a \propto \sqrt{T}$.

The exact relationship can be shown to be $a = \sqrt{\gamma \mathrm{R} T}$

where γ is the ratio of specific heats, C_p/C_v , and R is the gas constant.

For air, Υ = 1.40 and R = 287.05 J/(kg K).

Example: estimate the speed of sound in air at sea level, std atmosphere. *T* = 288.16 K.

$$a = \sqrt{1.40 \times 287.05 \times 288.16}\ \mathrm{m/s} = 340.30\ \mathrm{m/s}.$$

This is 1225 km/hr.

The fact that this speed is finite explains why you may be able to see a supersonic aircraft well before you hear it.

Typical fluid properties

Table 3.1 Values of the density and viscosity of several media, at normal conditions.

Medium	ρ (kg/m^3)	μ (Ns/m^2)	ν (m^2/s)	ρ/ρ_{air}	μ/μ_{air}	ν/ν_{air}
Water	998	1.00×10^{-3}	1.00×10^{-6}	825	54.9	6.67×10^{-2}
Air	1.21	1.82×10^{-5}	1.50×10^{-5}	1	1	1
Glycerine	1,260	1.50	1.19×10^{-3}	1,041	82,417	79.3
Lubricating oil	960	0.986	1.03×10^{-3}	793	54,176	68.7
Mercury	13,500	1.57×10^{-3}	1.16×10^{-7}	11,157	86.3	7.73×10^{-3}

Torenbeek & Wittenberg

Note that the group μ/ρ occurs so often in fluid mechanics that it is given its own name, kinematic viscosity, and with symbol υ. It is a kind of diffusivity coefficient.

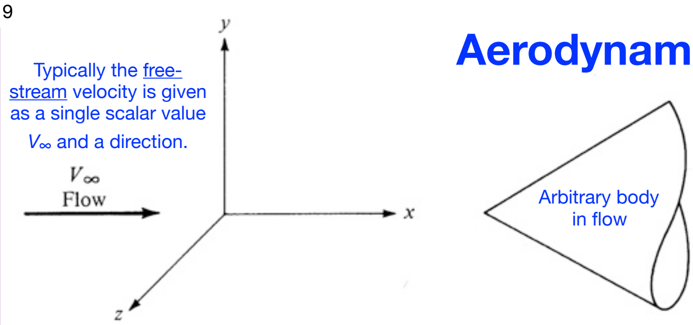
$$\nu = \mu/\rho$$

The viscosity *μ* is sometimes called dynamic viscosity to distinguish it from the kinematic viscosity.

Viscosity of air (curve fit): $\mu_{\mathrm{air}} = 1.458 \times 10^{-6} \dfrac{T^{3/2}}{T + 110.4}\ \mathrm{Pa.s} \qquad (T \text{ in K})$

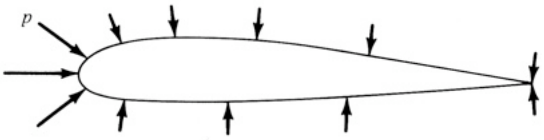
Density of air (gas law): $\rho = \dfrac{p}{\mathrm{R}T}$

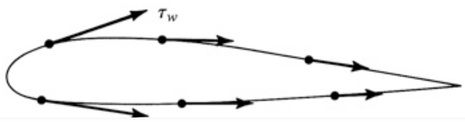
Aerodynamic forces



Anderson

$$\left.\begin{aligned} p &= p(x,y,z) \\ \rho &= \rho(x,y,z) \\ T &= T(x,y,z) \\ \mathbf{V} &= \mathbf{V}(x,y,z) \end{aligned}\right\} \text{Flow field}$$





For typical aircraft flight, almost all LIFT results from pressure differences, while DRAG is caused by pressure differences and tangential stresses in approximately equal measure.

The flow field as described by the variables p, ρ, T and $\boldsymbol{V}$ at all points (x,y,z) in space in a region of interest completely describes the flow.

(Secondary fluid variables like viscosity can be defined in terms of p, ρ, T).

The flow field is sufficient to define everything about the flow, including the forces it exerts on bodies immersed in it.

All the forces that are exerted by the fluid on objects in it result either from normal stress, i.e. pressure p, or tangential stress, τ.

Tangential stresses on solid bodies are produced *only* by viscosity and velocity gradients normal to the surface, while normal stresses can result *only* from pressure.

To obtain the total force exerted by the body, we integrate the surface tractions p and τ_w over the surface area of the body.

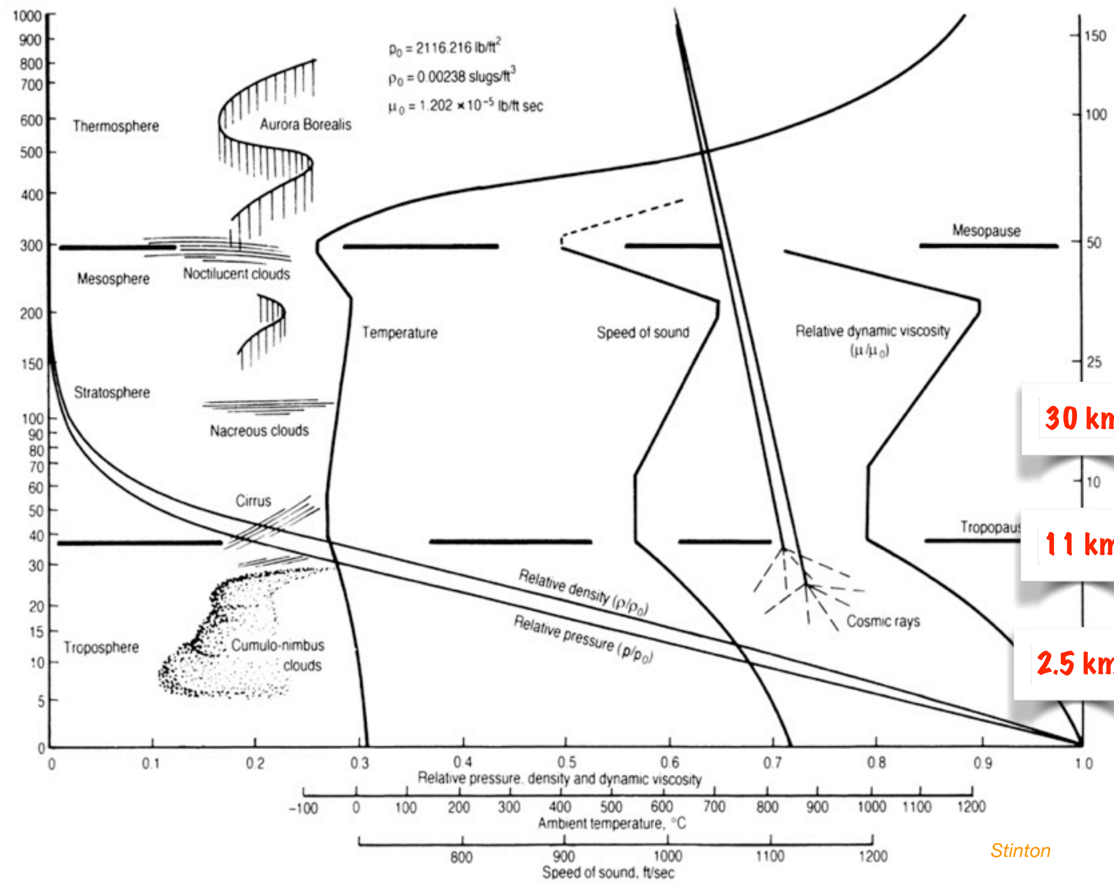
From Newton's 3rd law, the body exerts the same total amount of force back on the fluid.

The Earth's atmosphere, hydrostatics and the Standard Atmosphere model

Brandt el al. Ch 2

Torenbeek & Wittenberg Ch 2

International Standard Atmosphere



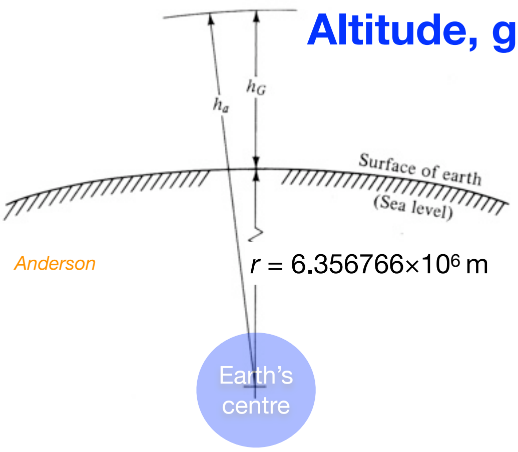
30 km — Maximum for airbreathing propulsion

11 km — Efficient fan jet cruise altitude

2.5 km — Limit without cabin pressurization

Stinton

Altitude, gravity, and hydrostatics



h_a = absolute altitude from the centre of the earth

h_G = geometric (or true) altitude above sea level

According to Newton's law of gravitational attraction, gravitational acceleration g falls with altitude

$$g = g_0 \left(\frac{r}{h_a}\right)^2 = g_0 \left(\frac{r}{r + h_G}\right)^2$$

$g_0 = 9.8065 \text{m/s}^2$ at the surface, and $r = 6.356766 \times 10^6$m.

<u>Pressure in a static fluid must decrease with altitude as the weight of fluid to be supported falls.</u>

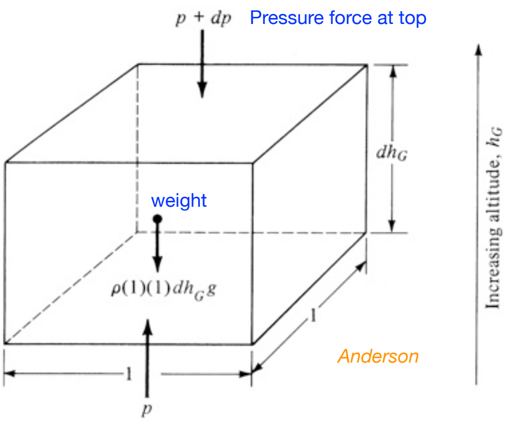
Consider a rectangular element with horizontal sides of unit length and an infinitesimal vertical height dh_G.

Pressure and weight forces must sum to zero.

$$p = p + dp + \rho g \, dh_G$$

or

$$dp = -\rho g \, dh_G$$



<u>This is the fundamental differential equation of hydrostatics.</u>

In principle we can integrate it to get pressure at any altitude.

Two slight problems:

1. g is a function of height
2. ρ is a function of height

Note that if g and ρ are constants, then $\Delta p = -\rho g \Delta h_G$.

Simple approximate solutions in hydrostatics

For small changes in elevation, neither gravity nor density variations are important and we can use

$$\Delta p \approx -\rho \mathrm{g} \Delta h$$

Example 1

How much does air pressure at the top of the Empire State Building (height = 381 m) differ from that at ground level, assuming ρ = 1.225 kg/m³ ?

$$\begin{aligned}\Delta p \approx -\rho \mathrm{g} \Delta h &= -1.225 \times 9.8065 \times 381 \text{ Pa} \\ &= -4825 \text{ Pa}\end{aligned}$$

This is $\left(\frac{4825}{101325} \times 100\right)\% = 4.5\%$ lower than at ground level.

Example 2

What is the absolute pressure at a depth of 20 m in seawater, ρ = 1025 kg/m³ ?

$$\begin{aligned}\Delta p \approx -\rho \mathrm{g} \Delta h &= -1025 \times 9.8065 \times -20 \text{ Pa} \\ &= 201{,}033\text{Pa}\end{aligned}$$

Absolute pressure $p = p_0 + \Delta p = 101{,}325 + 201{,}033$ Pa

$$= 302{,}358 \text{ Pa} = 302.4 \text{ kPa} \quad = \frac{302{,}358}{101{,}325} \text{ atm} = 2.98 \text{ atm}$$

Altitude, gravity, and hydrostatics

1. To deal with g being a function of height, we make a change of variables.

If we set g = g_0 (sea-level value) we can compensate by using a fictitious altitude instead of the (true) geometric altitude for the same change in pressure. This 'fake' height is called geopotential altitude and given the symbol *h*.

Instead of $\mathrm{d}p = -\rho \mathrm{g}\, \mathrm{d}h_G$ (g: variable) suppose $\mathrm{d}p = -\rho \mathrm{g}_0\, \mathrm{d}h$ (g_0: constant) Fine, except we can't yet define *h*.

Since $1 = \frac{\mathrm{g}_0}{\mathrm{g}} \frac{\mathrm{d}h}{\mathrm{d}h_G}$ we get $\mathrm{d}h = \frac{\mathrm{g}}{\mathrm{g}_0} \mathrm{d}h_G$ And we knew $\frac{\mathrm{g}}{\mathrm{g}_0} = \frac{r^2}{(r+h_G)^2}$ hence $\mathrm{d}h = \frac{r^2}{(r+h_G)^2} \mathrm{d}h_G$

Integrate: $\int_0^h \mathrm{d}h = \int_0^{h_G} \frac{r^2}{(r+h_G)^2} \mathrm{d}h_G = r^2 \int_0^{h_G} \frac{1}{(r+h_G)^2} \mathrm{d}h_G$

$$h = r^2 \left[\frac{-1}{r+h_G}\right]_0^{h_G} = r^2 \left(\frac{-1}{r+h_G} + \frac{1}{r}\right) = \frac{r}{r+h_G} h_G = \frac{1}{1+h_G/r} h_G$$

Now we know *h*, given h_G.

Recall *r* = 6.356766×10⁶m

So in the calculations to follow, whenever we are given the true (geometric) altitude h_G, we first calculate the geopotential altitude *h*.

Because *r* is huge, the geometric and geopotential altitudes are always quite similar.

E.g. at h_G = 9000m, *h* = 9000/(1+9000/6.356766×10⁶)m = 8987m.

With regard to our differential equation d*p* = –ρg dh_G, we achieve the same effect with d*p* = –ρg_0d*h*, provided we define the density-dependence with height in terms of *h* instead of h_G.

This is exactly what the Standard Atmosphere model does.

International Standard Atmosphere – 1

The ISA *defines* a set of linear variations of absolute temperatures with *geopotential* altitude h, as shown.

<u>2. To deal with ρ being a function of height we use the perfect gas law $\rho = p/RT$ to redefine density in terms of temperature.</u>

$$\mathrm{d}p = -\rho g_0 \,\mathrm{d}h = -\frac{p}{RT} g_0 \,\mathrm{d}h$$

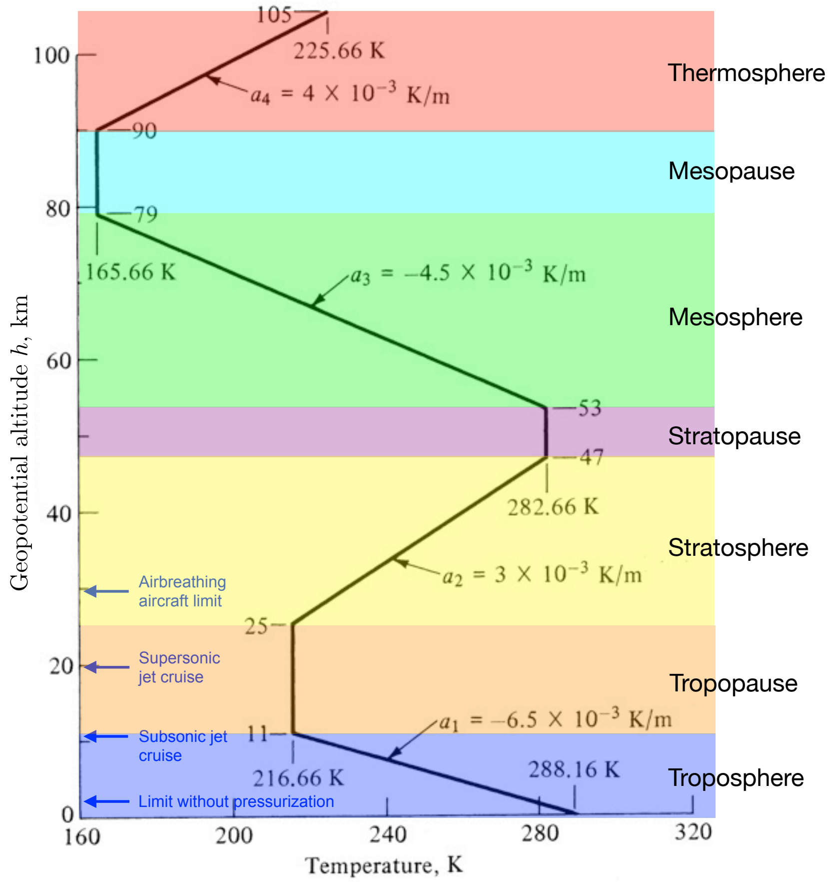
$$\frac{\mathrm{d}p}{p} = -\frac{g_0}{RT}\mathrm{d}h$$

$$\int_0^h \frac{\mathrm{d}p}{p} = -\frac{g_0}{R}\int_0^h \frac{\mathrm{d}h}{T}$$

Integrate to get pressure at any altitude. We know temperature, so then if we need density we re-use the perfect gas law again. Simple.

Note that either the temperature is constant with h, or it varies linearly with a *lapse rate* a_i.

Anderson



$$\frac{\mathrm{d}p}{p} = -\frac{g_0}{RT}\mathrm{d}h$$

International Standard Atmosphere – 2

McCormick

Case A. <u>Constant lapse rate layer</u>: $\mathrm{d}T/\mathrm{d}h = a$, with bottom temperature T_0, pressure p_0.

$$\frac{\mathrm{d}p}{p} = -\frac{g_0}{R\,\mathrm{d}T/\mathrm{d}h}\frac{\mathrm{d}T}{T} = -\frac{g_0}{aR}\frac{\mathrm{d}T}{T}$$

$$\ln\frac{p}{p_0} = -\frac{g_0}{aR}\ln\frac{T}{T_0}$$

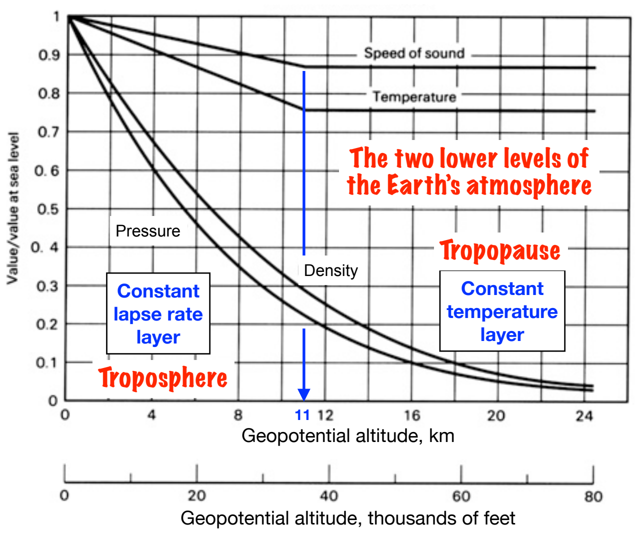
$$\frac{p}{p_0} = \left(\frac{T}{T_0}\right)^{-g_0/aR}$$ and $$\frac{\rho}{\rho_0} = \left(\frac{T}{T_0}\right)^{-(g_0/aR)-1}$$

Case B. <u>Constant temperature layer</u>, $T=T_c$ starting at geopotential altitude h_c where $p=p_c$.

$$\frac{\mathrm{d}p}{p} = -\frac{g_0}{RT_c}\mathrm{d}h$$

$$\ln\frac{p}{p_c} = -\frac{g_0}{RT_c}(h-h_c)$$

$$\frac{p}{p_c} = e^{(-g_0/RT_c)(h-h_c)}$$ and $$\frac{\rho}{\rho_c} = e^{(-g_0/RT_c)(h-h_c)}$$



<u>E.g. Find all properties at geometric altitude 9000m.</u> This is in the troposphere, case A, lapse rate -6.5×10^{-3} K/m.

T_0 = 288.16K, p_0 = 101325Pa, ρ_0 = 1.225kg/m³ and g_0 = 9.8065m/s². R = 287.1 J/(kg.K).

At h_G = 9000m, h = 9000/(1+9000/6.356766×10⁶)m = 8987m.

T = 288.16 - 8987×6.5×10⁻³K = 229.74K $\qquad T/T_0$ = 229.74/288.16 = 0.79727.

p = 101325 × $0.79727^{-9.8065/(-0.0065\times287.1)}$ Pa = 101325 × 0.30408 Pa = <u>30.81 kPa</u>.

ρ = 1.225 × $0.79727^{-9.8065/(-0.0065\times287.1)-1}$ kg/m³ = 1.225 × 0.38136 kg/m³ = <u>0.4672 kg/m³</u>.

<u>cf. Table</u>
30.80 kPa ✔
0.4671 kg/m³ ✔

Summary ISA table for geometric altitudes up to 20km

h_G(km)	T (°C)	a/a_0	$\delta = p/p_0$	$\sigma = \rho/\rho_0$
0	15.0	1.0000	1.0000	1.0000
1	8.5	0.9887	0.8870	0.9075
2	2.0	0.9772	0.7846	0.8217
3	-4.5	0.9656	0.6920	0.7423
4	-11.0	0.9538	0.6085	0.6689
5	-17.5	0.9420	0.5334	0.6012
6	-24.0	0.9299	0.4660	0.5389
7	-30.5	0.9178	0.4057	0.4817
8	-37.0	0.9054	0.3519	0.4292
9	-43.5	0.8929	0.3040	0.3813
10	-50.0	0.8802	0.2615	0.3376
11	-56.5	0.8674	0.2240	0.2978
12	-56.5	0.8671	0.1909	0.2541
13	-56.5	0.8671	0.1632	0.2171
14	-56.5	0.8671	0.1395	0.1856
15	-56.5	0.8671	0.1192	0.1581
16	-56.5	0.8671	0.1019	0.1356
17	-56.5	0.8671	0.0871	0.1159
18	-56.5	0.8671	0.0744	0.0990
19	-56.5	0.8671	0.0636	0.0847
20	-56.5	0.8671	0.0544	0.0724

a_0	340.3 m/s
p_0	101,325 Pa
ρ_0	1.225 kg/m^3
T_0	288.15 K
g_0	9.8065 m/s^2
R_{air}	287.16 J/kg K
γ_{air}	1.40

Interpolation in tables

h_G (km)	T (°C)	a/a_0	$\delta = p/p_0$	$\sigma = \rho/\rho_0$
0	15.0	1.0000	1.0000	1.0000
1	8.5	0.9887	0.8870	0.9075
2	2.0	0.9772	0.7846	0.8217
3	-4.5	0.9656	0.6920	0.7423
4	-11.0	0.9538	0.6085	0.6689
5	-17.5	0.9420	0.5334	0.6012
6	-24.0	0.9299	0.4660	0.5389
7	-30.5	0.9178	0.4057	0.4817
8	-37.0	0.9054	0.3519	0.4292
9	-43.5	0.8929	0.3040	0.3813
10	-50.0	0.8802	0.2615	0.3376
11	-56.5	0.8674	0.2240	0.2978
12	-56.5	0.8671	0.1909	0.2541
13	-56.5	0.8671	0.1632	0.2171
14	-56.5	0.8671	0.1395	0.1856
15	-56.5	0.8671	0.1192	0.1581
16	-56.5	0.8671	0.1019	0.1356
17	-56.5	0.8671	0.0871	0.1159
18	-56.5	0.8671	0.0744	0.0990
19	-56.5	0.8671	0.0636	0.0847
20	-56.5	0.8671	0.0544	0.0724

a_0	340.3 m/s
p_0	101,325 Pa
ρ_0	1.225 kg/m^3
T_0	288.15 K
g_0	9.8065 m/s^2
R_{air}	287.16 J/kg K
γ_{air}	1.40

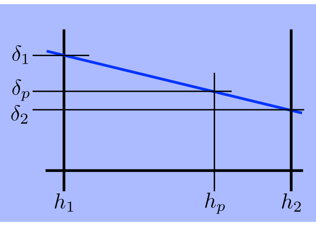
<u>Two standard tasks with table interpolation:</u>

1. Given the independent variable (here, altitude, h_G), estimate a dependent variable (e.g. T);
2. Given a dependent variable (e.g. T), estimate the independent variable (h_G).

In either case, use linear interpolation, based on similar triangles. It's usually helpful to draw a diagram as a visual aid.

In the example below, we are given density ratio δ_p and want to know the altitude h_p at which it occurs.

First, identify the bracketing values δ_1 and δ_2 and corresponding h_1 and h_2, then work out a formula based on similar triangles.



Similar triangles:

$$\frac{\delta_1 - \delta_p}{h_p - h_1} = \frac{\delta_1 - \delta_2}{h_2 - h_1}$$

rearrange to solve for h_p:

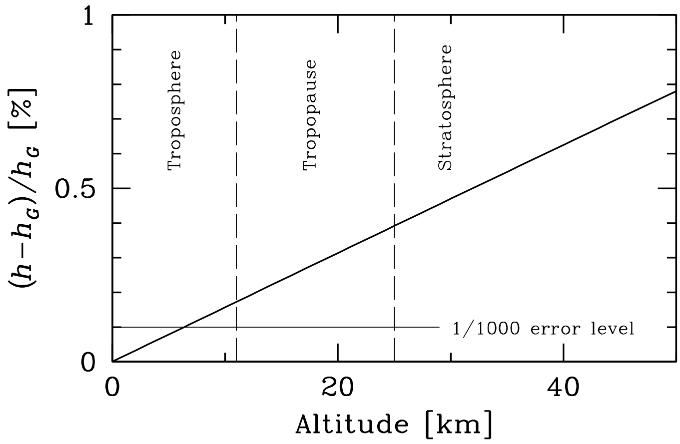
$$h_p = h_1 + \frac{\delta_1 - \delta_p}{\delta_1 - \delta_2}(h_2 - h_1)$$

(simplified) International Standard Atmosphere

The difference between the true and geopotential altitudes is quite small at typical operational heights (e.g. 0–11 km).

Thus one can usually use them interchangeably in calculations (i.e. assume g is a constant) and this is often done.

If we re-do the previous calculations using just the geometric altitude then we will find that the differences are small.



Re-estimate properties at geometric altitude 9000m assuming this also is the geopotential altitude. Tropopause, case A, lapse rate -6.5×10^{-3} K/m.

T_0 = 288.16 K, p_0 = 101325 Pa, ρ_0 = 1.225 kg/m³ and g_0 = 9.8065 m/s². R = 287.1 J/(kg.K).

T = 288.16 - 9000×6.5×10⁻³K = 229.66 K (cf. previous 229.74 K)

T/T_0 = 229.66/288.16 = 0.79699 (cf. 0.79727).

$p = 101325 \times 0.79699^{-9.8065/(-0.0065\times287.1)}$ Pa = 30.76 kPa (cf. 30.81 kPa, 0.16% error)

$\rho = 1.225 \times 0.79699^{-9.8065/(-0.0065\times287.1)\ -\ 1}$ kg/m³ = 0.4665 kPa (cf. 0.4672 kg/m³, 0.15% error)

cf. Table
30.80 kPa
0.4671 kg/m³

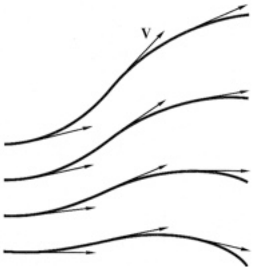
The temperatures, pressures and densities estimated using this approximation are always a little low, but often within 'engineering error' tolerance.

Basic fluid mechanics

Anderson Ch 4

Torenbeek & Wittenberg Ch 3

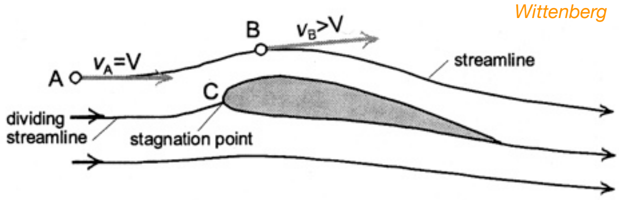
Introductory concepts



White

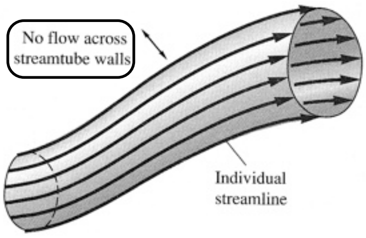
Streamlines are locally tangent to the velocity vector field.

By definition flow cannot cross a streamline.

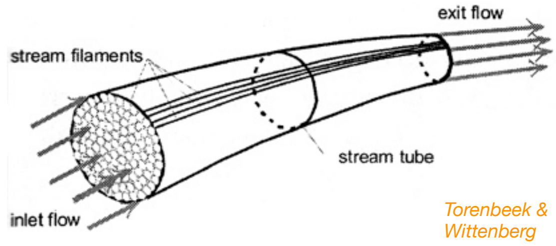


Torenbeek & Wittenberg

Streamlines cannot intersect except at **stagnation points** where the velocity is zero (typically, though, stagnation points are located on solid walls where a streamline attaches to the wall).



A **streamtube** is a collection of streamlines that pass through a simple closed curve.



Torenbeek & Wittenberg

Stream filaments are small stream tubes in which the flow properties may be taken as locally one-dimensional (1D) – only varying along the length of the tube.

Initial derivations assume flow is:

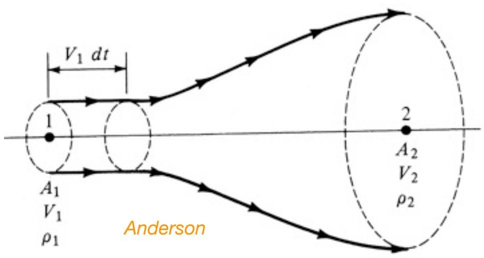
- 1-dimensional stream filaments ≡ stream tubes
- Steady
- Inviscid/frictionless
- Incompressible

3 fundamental conservation laws:

- Conservation of mass
- Conservation of momentum
- Conservation of energy

Conservation of mass: continuity equation

Physical principle: mass is conserved, i.e. neither created nor destroyed.



Anderson

Consider 1D steady flow in a stream tube with entry area A_1. Say the area-average flow speed and density here are V_1 and ρ_1.

The amount of volume swept out by the flow through this area in an small amount of time dt is V_1A_1dt.

The amount of mass carried by the flow through this area in an small amount of time dt is dm = $\rho_1V_1A_1$dt.

The mass flow rate through area A_1 is then $\frac{\mathrm{d}m}{\mathrm{d}t} = \dot{m}_1 = \rho_1 V_1 A_1$

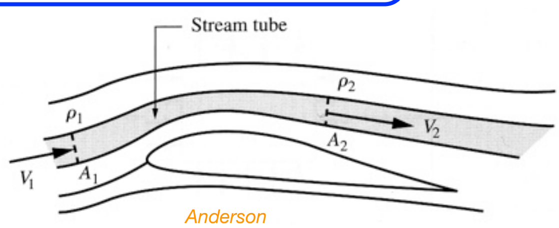
Further downstream, the cross-sectional area of the streamtube is A_2.

Since (1) the flow is steady, (2) no flow can cross the boundaries of the streamtube, and (3) mass has to be conserved, mass has to be transported across area A_2 at the same rate it was transported across A_1.

$\dot{m}_1 = \dot{m}_2 = \text{const.}$ Hence $\rho_1 V_1 A_1 = \rho_2 V_2 A_2 = \text{const.}$ **1D steady continuity equation.**

Note that if the flow is incompressible (so density is also constant), $V_1 A_1 = V_2 A_2$

In general: **where flow speeds up, streamtubes get narrower.**



Anderson

FLUID MECHANICS

Continuity example – Venturi tube

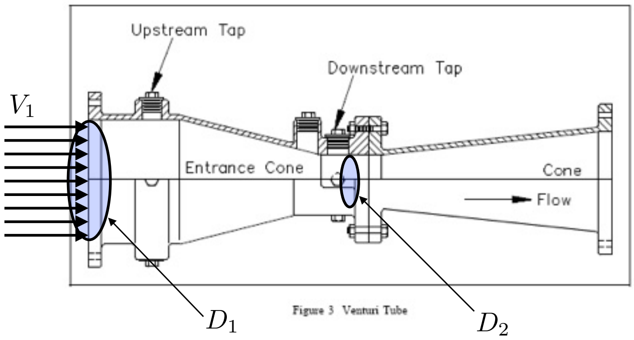


Figure 3 Venturi Tube

Can reasonably assume 1D flow provided boundary layers are thin.

Suppose D_1= 100 mm, D_2 = 40mm, V_1 = 0.5 m/s and this is a flow of water, ρ = 998 kg/m³.

Can safely assume flow is incompressible: $\rho_1 = \rho_2 = \rho$.

1. What is the mass flow rate?

$$\dot{m} = \rho A_1 V_1 = \rho \frac{\pi D_1^2}{4} V_1$$

$$= 998 \times \frac{\pi \times 0.1^2}{4} \times 0.5\,\mathrm{kg/s} \;= 998 \times 7.854 \times 10^{-3} \times 0.5\,\mathrm{kg/s} \;= 3.919\,\mathrm{kg/s}$$

2. What is the volume flow rate?

$$Q = \frac{\dot{m}}{\rho} = A_1 V_1 = A_2 V_2 \quad = 7.854 \times 10^{-3} \times 0.5\,\mathrm{m^3/s} \;= 3.927 \times 10^{-3}\,\mathrm{m^3/s}$$

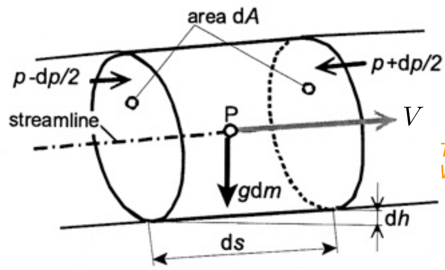
3. What is the speed V_2?

$$A_1 V_1 = A_2 V_2 \quad \text{or} \quad V_2 = \frac{A_1}{A_2} V_1 \;= \frac{D_1^2}{D_2^2} V_1 \;= \frac{0.1^2}{0.04^2} \times 0.5\,\mathrm{m/s} = 6.25 \times 0.5\,\mathrm{m/s} \;= 3.125\,\mathrm{m/s}$$

FLUID MECHANICS

Conservation of momentum: Euler's equation

Newton's 2nd law applied to steady frictionless flow along a stream filament.



Torenbeek & Wittenberg

Pressure force, LH end $\left(p - \frac{\mathrm{d}p}{\mathrm{d}s}\mathrm{d}s/2\right)\mathrm{d}A = (p - \mathrm{d}p/2)\,\mathrm{d}A$

RH end $(p + \mathrm{d}p/2)\,\mathrm{d}A$

Differential mass $\mathrm{d}m = \rho\,\mathrm{d}A\,\mathrm{d}s$

Component of weight force along streamline $-\mathrm{g}\,\mathrm{d}m\frac{\mathrm{d}h}{\mathrm{d}s} = -\mathrm{g}\rho\,\mathrm{d}A\,\mathrm{d}s\frac{\mathrm{d}h}{\mathrm{d}s} = -\rho\mathrm{g}\,\mathrm{d}h\,\mathrm{d}A$

Forces along streamline $\mathrm{d}F = (p - \mathrm{d}p/2)\mathrm{d}A - (p + \mathrm{d}p/2)\mathrm{d}A - \rho\mathrm{g}\,\mathrm{d}h\,\mathrm{d}A = -\mathrm{d}p\,\mathrm{d}A - \rho\mathrm{g}\,\mathrm{d}h\,\mathrm{d}A$

Newton's 2nd law $\mathrm{d}F = \mathrm{d}m\frac{\mathrm{d}V}{\mathrm{d}t}$

'Convective acceleration'

Acceleration of mass owing to change of speed along streamline $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = V\frac{\mathrm{d}V}{\mathrm{d}s}$

NB: even for time-steady flow!

Newton 2 $\mathrm{d}F = \mathrm{d}m\frac{\mathrm{d}V}{\mathrm{d}t} = \rho\,\mathrm{d}A\,\mathrm{d}s\,V\frac{\mathrm{d}V}{\mathrm{d}s} = -\mathrm{d}p\,\mathrm{d}A - \rho\mathrm{g}\,\mathrm{d}h\,\mathrm{d}A$ or $\rho\,\mathrm{d}A\,\mathrm{d}s\,V\frac{\mathrm{d}V}{\mathrm{d}s} = -\mathrm{d}p\,\mathrm{d}A - \rho\mathrm{g}\,\mathrm{d}h\,\mathrm{d}A$

or $\rho\,\mathrm{d}s\,V\frac{\mathrm{d}V}{\mathrm{d}s} = -\mathrm{d}p - \rho\mathrm{g}\,\mathrm{d}h$ or $\mathrm{d}p + \rho V\,\mathrm{d}V + \rho\mathrm{g}\,\mathrm{d}h = 0$ or $\frac{\mathrm{d}p}{\rho} + V\,\mathrm{d}V + \mathrm{g}\,\mathrm{d}h = 0$

Neglecting weight forces: $\mathrm{d}p = -\rho V\,\mathrm{d}V$

Euler's equation

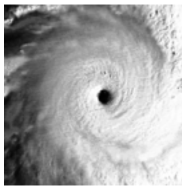
Increase in speed ⇒ fall in pressure.

For non-flowing fluid: $\mathrm{d}p = -\rho\mathrm{g}\,\mathrm{d}h$

Equation of hydrostatics

Increase in height ⇒ fall in pressure.

Normal pressure gradient and flow curvature



Newton's 2nd law applied to steady frictionless flow but NORMAL to a streamline.

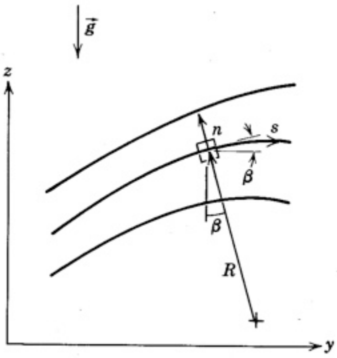
Now we consider curvature of streamlines.

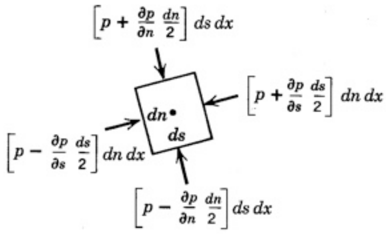
Take *s* and *n* as orthogonal coordinates along and normal to a streamline in a 2D flow.

Formally, since the flow is 2D, we have to allow partial (∂) derivatives of pressure but this is not central to the development.

We take *x* as the out-of-plane direction.

The local streamline radius of curvature is *R*.





Fox & McDonald

The flow-normal component of acceleration (centripetal) $a_n = -\dfrac{V^2}{R}$

Differential mass $\quad \mathrm{d}m = \rho\,\mathrm{d}n\,\mathrm{d}s\,\mathrm{d}x$

Forces normal to streamline

$$\mathrm{d}F = \left(p - \frac{\partial p}{\partial n}\frac{\mathrm{d}n}{2}\right)\mathrm{d}s\,\mathrm{d}x - \left(p + \frac{\partial p}{\partial n}\frac{\mathrm{d}n}{2}\right)\mathrm{d}s\,\mathrm{d}x - \rho \mathrm{g}\,\cos\beta\,\mathrm{d}n\,\mathrm{d}s\,\mathrm{d}x$$

$$= -\frac{\partial p}{\partial n}\mathrm{d}n\,\mathrm{d}s\,\mathrm{d}x - \rho \mathrm{g}\,\cos\beta\,\mathrm{d}n\,\mathrm{d}s\,\mathrm{d}x$$

Newton's 2nd Law

$$\mathrm{d}F = \mathrm{d}m\,a_n = -\rho\,\mathrm{d}n\,\mathrm{d}s\,\mathrm{d}x\frac{V^2}{R}$$

Rearrange

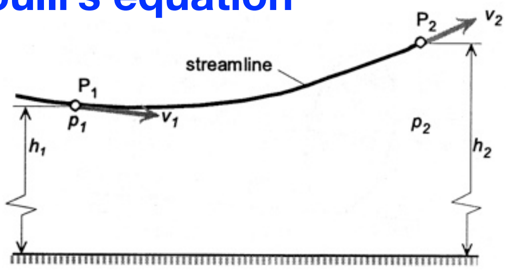
$$\frac{1}{\rho}\frac{\partial p}{\partial n} + \mathrm{g}\cos\beta = \frac{V^2}{R}$$

Or, since $\cos\beta = \dfrac{\partial z}{\partial n}$, $\quad \dfrac{1}{\rho}\dfrac{\partial p}{\partial n} + \mathrm{g}\dfrac{\partial z}{\partial n} = \dfrac{V^2}{R}$

Neglecting weight forces: $\quad \dfrac{\partial p}{\partial n} = \rho\dfrac{V^2}{R}$

1. Pressure decreases towards centre of curvature.
2. Pressure constant across flow if no curvature.

Conservation of energy: Bernoulli's equation



Torenbeek & Wittenberg

Euler's equation for steady flow $\quad \dfrac{\mathrm{d}p}{\rho} + V\,\mathrm{d}V + \mathrm{g}\,\mathrm{d}h = 0$

Integrate along streamline

$$\int_{p_1}^{p_2}\frac{\mathrm{d}p}{\rho} + \int_{V_1}^{V_2} V\,\mathrm{d}V + \int_{h_1}^{h_2}\mathrm{g}\,\mathrm{d}h = 0$$

or

$$\int_{p_1}^{p_2}\frac{\mathrm{d}p}{\rho} + \frac{1}{2}(V_2^2 - V_1^2) + \mathrm{g}(h_2 - h_1) = 0$$

Constant density case:

$$\frac{1}{\rho}(p_2 - p_1) + \frac{1}{2}(V_2^2 - V_1^2) + \mathrm{g}(h_2 - h_1) = 0$$

or $\quad p_1 + \tfrac{1}{2}\rho V_1^2 + \rho \mathrm{g}h_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \rho \mathrm{g}h_2$

or $\quad p + \tfrac{1}{2}\rho V^2 + \rho \mathrm{g}h = \mathrm{const.}$ **Bernoulli's equation**

Neglecting weight forces: $\quad p + \tfrac{1}{2}\rho V^2 = \mathrm{const.}$ Typical in aerospace applications to gas flows.

As pressure rises along a streamline in an incompressible frictionless flow, velocity must fall (and vice versa).

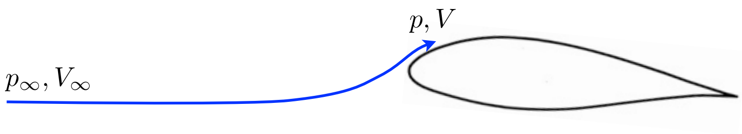
All terms have dimensions of energy per unit volume. **This is a statement of conservation of energy.** The pressure represents stored potential energy of compression in the fluid, the ½ρV^2 terms represent kinetic energy per unit volume, the ρgz terms represent gravitational potential energy.

Example application of Bernoulli's equation

An aircraft is flying at an altitude of h_G = 10km with a speed V_∞ = 150 m/s.

If at a point just above the top surface of the wing, the flow speeds up above the flight speed by 20%, what is (a) the local pressure relative to the surrounding air and (b) the absolute pressure?

Assume incompressible flow and neglect any elevation change.



$h_G = 10\ \text{km} \implies \sigma = 0.3376, \delta = 0.2615.$ (tables)

Hence $\rho = 0.3376 \times 1.225 = 0.41356\,\text{kg/m}^3, \quad p_\infty = 0.2615 \times 101\,325 = 26\,496\,\text{Pa}$

(Bernoulli) $p - p_\infty = \frac{1}{2}\rho V_\infty^2(1 - 1.44) = -2047\,\text{Pa}$

$p_{\text{abs}} = 26\,496 - 2047 = 24\,449\,\text{Pa}$

Ans (a): –2047 Pa,
(b): 24449 Pa.

Supposing the air had gained an additional altitude Δh = 1 m in rising above the wing.
How much additional pressure drop would that cause?

(hydrostatics) $\Delta p \approx \rho g \Delta h = -0.41356 \times 9.8065 \times 1 = -4\,\text{Pa}$

Ans: –4 Pa.

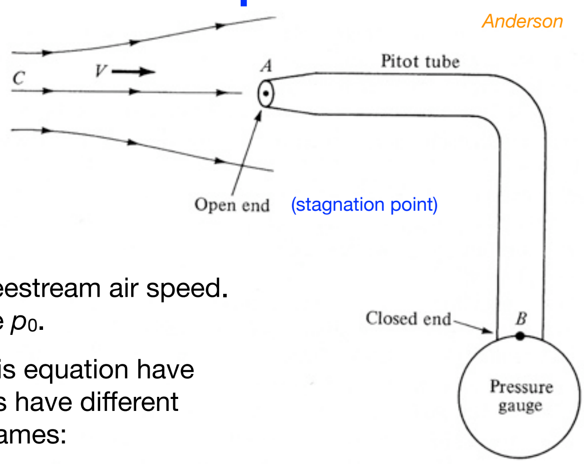
Velocity measurement via Bernoulli's equation

Measurement of air speed

We note that if a fluid can be brought to rest without friction losses (say along a streamline that attaches to the leading edge of a body at a *stagnation point*) then Bernoulli's equation says that for incompressible flow

$p + \frac{1}{2}\rho V^2 = p_0$ where p_0 is called the *total pressure*.

Anderson



This relationship is very commonly used to estimate the freestream air speed. A *Pitot tube* and a pressure gauge can be used to evaluate p_0.

Before examining this in detail, note that all the terms in this equation have dimensions of pressure. By convention, the different terms have different names. We just saw the *total pressure.* Here are all the names:

$$p + \frac{1}{2}\rho V^2 = p_0$$

static pressure + dynamic pressure = total pressure


Pitot tube mounted on a wing

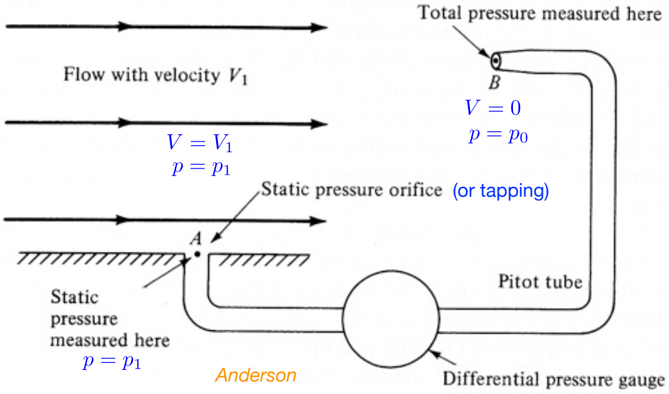
Paradoxically the name *static pressure* does not imply a lack of motion – which is associated only with the total pressure (a.k.a. stagnation pressure).

The name *dynamic pressure* is a good one.
Often the symbol q is used as shorthand.

$$q \equiv \frac{1}{2}\rho V^2$$

Now supposing we can simultaneously measure p and p_0. Then if we know ρ we can estimate V.

Applications of Bernoulli's equation

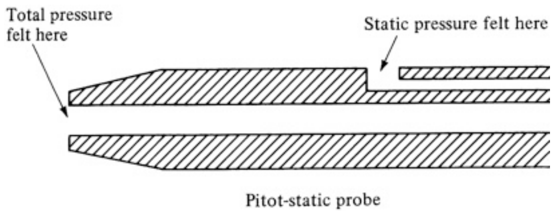


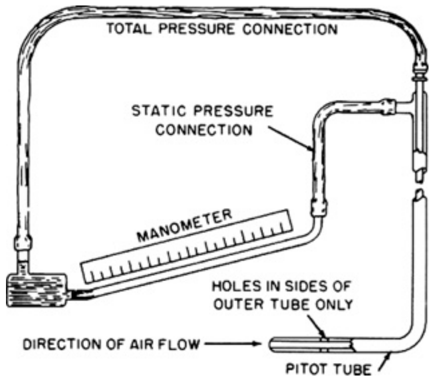
Here is one means to do this. If we also measure the pressure on a wall in a region where there is no streamline curvature, then we get a good estimate of the static pressure in the free-stream, e.g. p_1, and can then find the velocity there, say V_1.

$$p_1 + \tfrac{1}{2}\rho V_1^2 = p_0 \quad \text{or} \quad V_1 = \sqrt{\frac{2(p_0 - p_1)}{\rho}}$$

(Note we need density ρ, as well – if we measure T we could estimate it from the static pressure and the perfect gas law.)

Pressure tappings to obtain both the total and static pressure can be combined in one device, the *Pitot-static probe* (or Pitot-static tube). There are two input ports, which can be connected to a differential pressure gauge (e.g. a manometer).

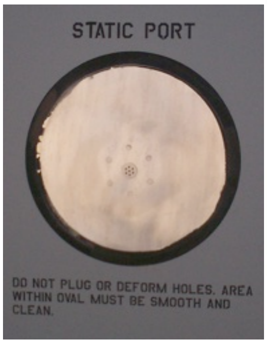


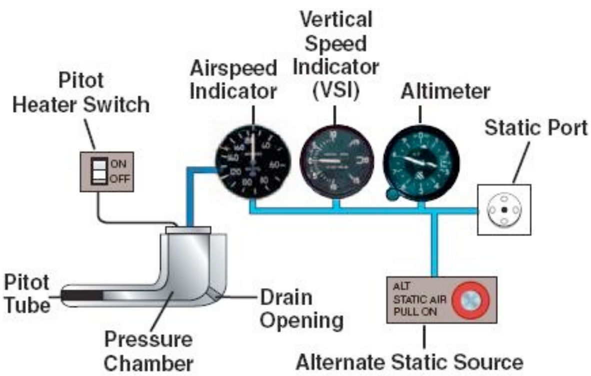


Aircraft hardware related to Bernoulli's equation

Typically aircraft use separate pitot and static tappings. Often the static tapping is somewhere on the fuselage, while the pitot tube may be on the wing, front of the fuselage, or elsewhere. The integrity of both tappings, and the associated tubes, is critical for safe flying.

Note that the static tapping may be connected to a number of devices as well as the airspeed indicator (e.g. to estimate altitude, vertical speed, and, in combination with a temperature measurement could be used to estimate the local air density).





Crash of Air France Flight 447, 2009, thought to caused by icing of pitot tube. 228 passengers and crew lost.

TAS and IAS (a.k.a. EAS)

Static pressure + dynamic pressure = stagnation pressure

$$p + \tfrac{1}{2}\rho V^2 = p_0 \qquad p + q = p_0 \qquad p_0 - p = q = \tfrac{1}{2}\rho V^2$$

Now $p_0 - p$ is the difference obtained between the two pressure tappings of a Pitot-static combination.

If we want to estimate the true speed V we also have to know the density, ρ.

If ρ is *assumed* to be the ISA sea-level value, then the *equivalent air speed* V_e is obtained.

In an aircraft this is what is shown by the *air speed indicator* instrument.

The equivalent air speed is also known as the *indicated air speed* (IAS).

This is typically lower than the *true air speed* (TAS) but is very useful to the pilot at all altitudes because it is always directly related to the dynamic pressure.

E.g. in level flight an aircraft always stalls at the same IAS, regardless of altitude and TAS.

$$p - p_0 = q = \tfrac{1}{2}\rho V^2 = \tfrac{1}{2}\rho_{SL} V_e^2 \qquad V = \sqrt{\frac{2q}{\rho}} = \sqrt{\frac{2(p_0 - p)}{\rho}}$$

$$\frac{\rho_{SL} V_e^2}{\rho V^2} = 1 \qquad V_e = \sqrt{\frac{2q}{\rho_{SL}}} = \sqrt{\frac{2(p_0 - p)}{\rho_{SL}}}$$

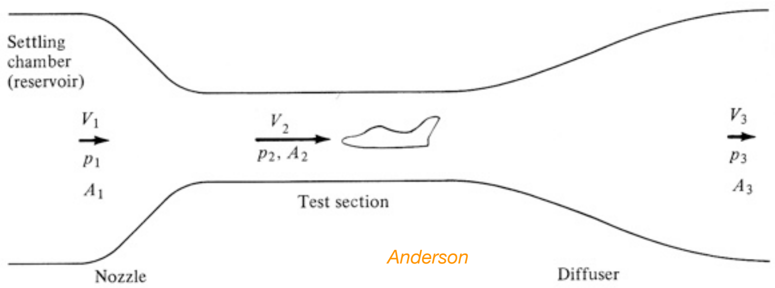
$$\frac{V_e}{V} = \frac{\text{IAS}}{\text{TAS}} = \sqrt{\frac{\rho}{\rho_{SL}}} = \sqrt{\sigma}$$

Applications of continuity AND Bernoulli equations

Recall: $\rho_1 V_1 A_1 = \rho_2 V_2 A_2$ and $p_1 + \tfrac{1}{2}\rho V_1^2 = p_2 + \tfrac{1}{2}\rho V_2^2$

$$p + \tfrac{1}{2}\rho V^2 = \text{constant along a streamline}$$

Low speed (subsonic) wind tunnel, or flow in a duct



If the velocities are low (say less than 50% the speed of sound, 340.3m/s at sea level) then density remains nearly constant. Also we work with average flow speeds at any location along the duct.

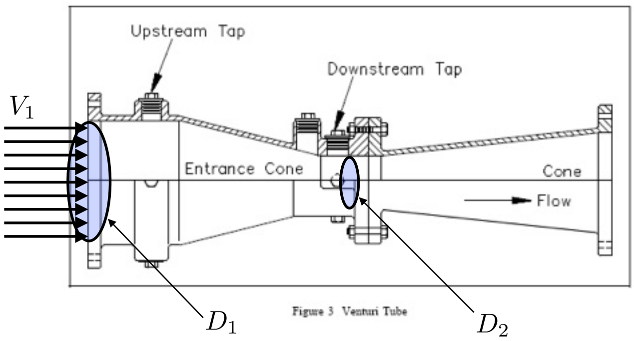
Then from the continuity equation $V_1 A_1 = V_2 A_2 = V_3 A_3$ and e.g. $V_2 = \dfrac{A_1}{A_2} V_1$

Bernoulli's equation tells us $p_1 + \tfrac{1}{2}\rho V_1^2 = p_2 + \tfrac{1}{2}\rho V_2^2 = p_3 + \tfrac{1}{2}\rho V_3^2$ and $V_2^2 = \dfrac{2}{\rho}(p_1 - p_2) + V_1^2$

Now using continuity, eliminate V_1: $V_2^2 = \dfrac{2}{\rho}(p_1 - p_2) + \left(\dfrac{A_2}{A_1}\right)^2 V_2^2$ Or $V_2 = \sqrt{\dfrac{2(p_1 - p_2)}{\rho[1 - (A_2/A_1)^2]}}$

I.e. we can control the flow speed by influencing ($p_1 - p_2$), or estimate V_2 if we know ($p_1 - p_2$).

Example application of continuity and Bernoulli equations



Suppose D_1= 100 mm, D_2 = 40mm, V_1 = 0.5 m/s and this is a flow of water, ρ = 998 kg/m³.

Can safely assume flow is incompressible: $\rho_1 = \rho_2 = \rho$.

What is the pressure difference between the upstream and the downstream pressure tappings?

We have already worked out from continuity that V_2 = 3.125 m/s.

Bernoulli $\quad p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2 \quad$ i.e. $\quad p_1 - p_2 = \frac{1}{2}\rho(V_2^2 - V_1^2)$

$$= 0.5 \times 998 \times (3.125^2 - 0.5^2)\ \mathrm{Pa}$$

$$= 0.5 \times 998 \times 9.5156\ \mathrm{Pa}$$

$$= 4748\ \mathrm{Pa}$$

$$= 4.748\ \mathrm{kPa}$$

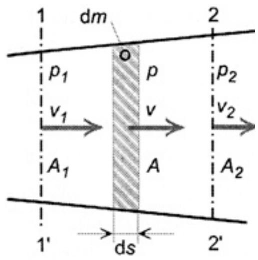
Integral form of momentum equation

Steady 1D flow in a stream filament.

Newton's 2nd law $\mathrm{d}F = \mathrm{d}m\dfrac{\mathrm{d}V}{\mathrm{d}t}$

$= \rho A\,\mathrm{d}s\dfrac{\mathrm{d}V}{\mathrm{d}t} = \rho A\,V\,\mathrm{d}V = \dot{m}\,\mathrm{d}V$

where $\dot{m} = \mathrm{const.}$ (from continuity)



$$F = \int \mathrm{d}F = F_{\mathrm{ext}} + F_{\mathrm{int}} = \dot{m}\int \mathrm{d}V = \dot{m}(V_2 - V_1)$$

Assuming internal forces (e.g. gravity) are zero:

$$F_{\mathrm{ext}} = \dot{m}(V_2 - V_1)$$

Say F_{ext} is caused by pressure differences on faces 1 and 2, as well as any other externally applied forces.

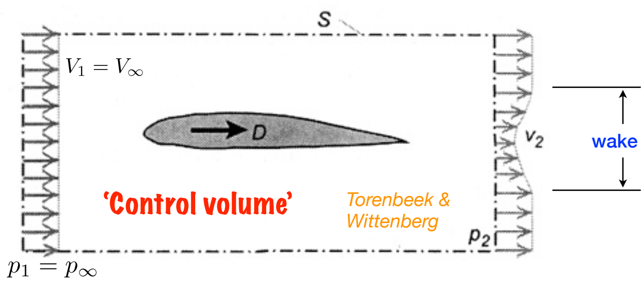
$$F_{\mathrm{ext}} = p_1 A_1 - p_2 A_2 + R_x = \dot{m}(V_2 - V_1)$$

Steady flow in a larger domain with an immersed body on which drag force *D* is exerted.

Domain is large enough that on the upper/lower far-field boundaries the flow is undisturbed ⇒ τ = 0 there.

Then (Newton's 3rd law) $D \equiv -R_x$

Say, on the upstream boundary $p_1 = p_\infty,\ \ V_1 = V_\infty$



Also, the downstream boundary far enough away that $p_2 = p_\infty$

Then $D = \dot{m}(V_\infty - V_2)$ However, the development above is for 1D flow. If V_2 non-uniform on *S* (integrate):

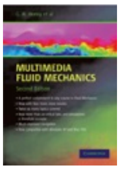
$$D = \int_S \rho V(V_\infty - V)\,\mathrm{d}S$$

This method is often used to estimate drag.

momentum deficit – only significant in wake of body where $V \neq V_\infty$

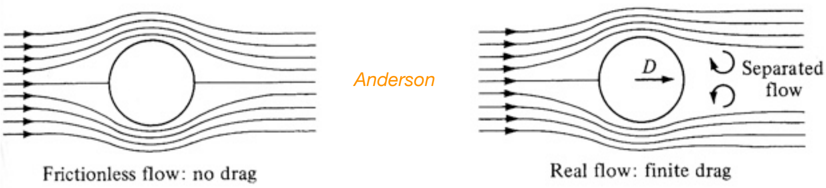
Introduction to viscous laminar and turbulent flow

Torenbeek & Wittenberg Ch 3
Anderson Ch 5

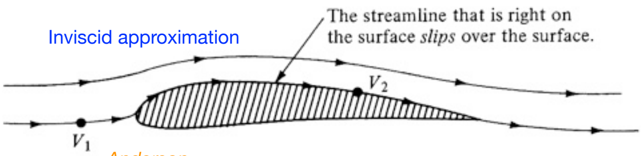


Introduction

All flows of real fluids (with viscosity) show viscous (friction) effects. Exactly how these are manifested depends on the flow in question. In some flows the effects can be profound and completely change the pressure distribution.

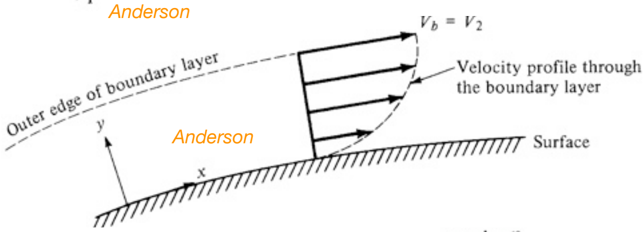


However, typically in aeronautics, basic shapes are streamlined (i.e. streamlines follow the shape) and in these cases, inviscid theory (e.g. Euler, Bernoulli equations) does a good job of predicting flow speeds close to, and pressure acting on, the surfaces involved.



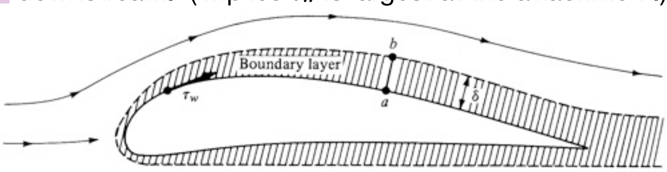
In these cases, the effects of viscosity are mostly confined to thin boundary layers, where the flow speed drops rapidly to zero at the solid surface. This leads to viscous shear stress on the wall, according to

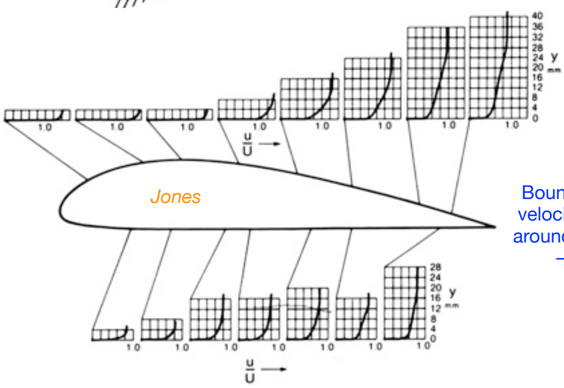
$$\tau_w = \mu \left(\frac{\mathrm{d}V}{\mathrm{d}y} \right)_{y=0}$$



<u>The resulting skin friction is the dominant source of drag for jet transport aircraft.</u>

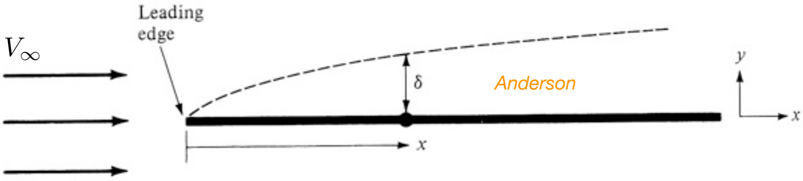
Boundary layers start out very thin where flow attaches to the body, and their thickness δ generally increases downstream. (Implies τ_w is largest at the attachment.)





Boundary layer velocity profiles around an airfoil – flight test data.

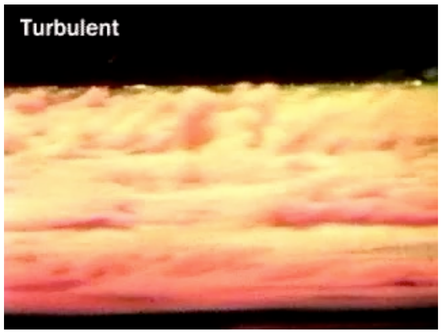
Boundary layer flows, laminar and turbulent



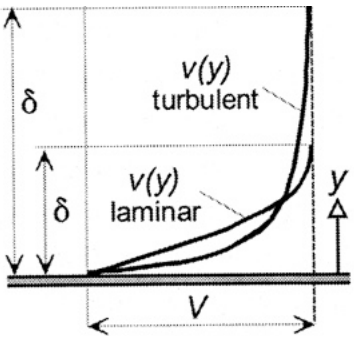
The thickness δ of a boundary layer on a flat plate increases with streamwise distance x in a way that scales from one flow to another according to the local Reynolds number Re_x:

$$Re_x = \frac{\rho V_\infty x}{\mu}$$

Proceeding along the plate, an originally well-ordered (laminar) flow first becomes unstable and then, turbulent.



δ is taken as the height at which 99% of freestream speed is reached.



Torenbeek & Wittenberg

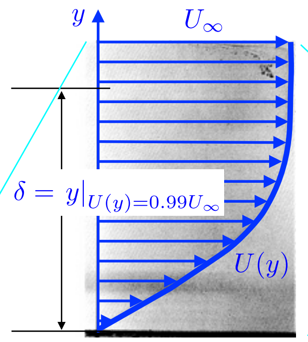
A turbulent boundary layer is better at mixing high-velocity fluid towards the wall, making the flow profile 'fuller' and increasing its gradient at the wall.

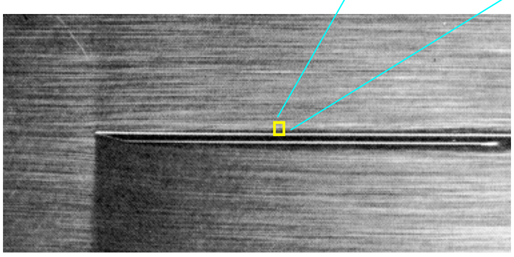
Given equal δ, V_∞: $$\left(\frac{\mathrm{d}V}{\mathrm{d}y}\right)_{y=0,\,\mathrm{laminar}} < \left(\frac{\mathrm{d}V}{\mathrm{d}y}\right)_{y=0,\,\mathrm{turbulent}}$$

Hence $\tau_{w\,\mathrm{laminar}} < \tau_{w,\,\mathrm{turbulent}}$ Turbulent BLs produce more drag.

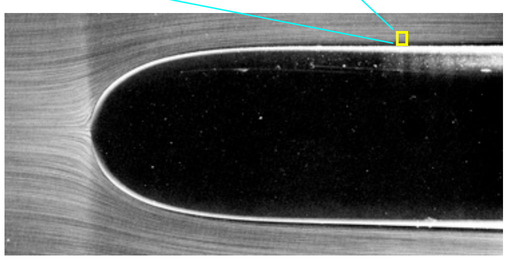
Laminar boundary layers

Pulse of dye released from wire passing through boundary layer shows the velocity profile





Laminar flow past a flat plate



Laminar flow past a round-nosed body

Flow is smooth and orderly.

Cross-flow communication between different layers parallel to wall occurs via molecular diffusion.

We can directly solve the equations of motion in steady flow to find velocities, thicknesses, etc.

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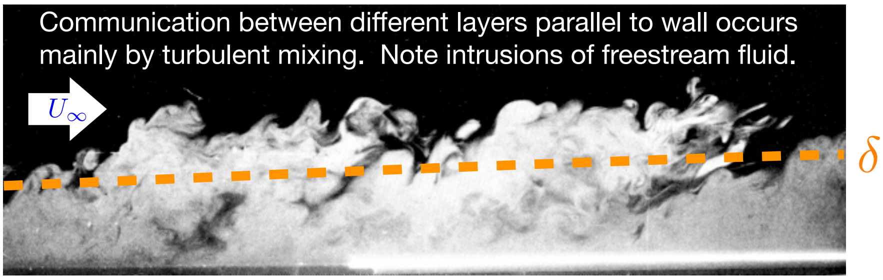
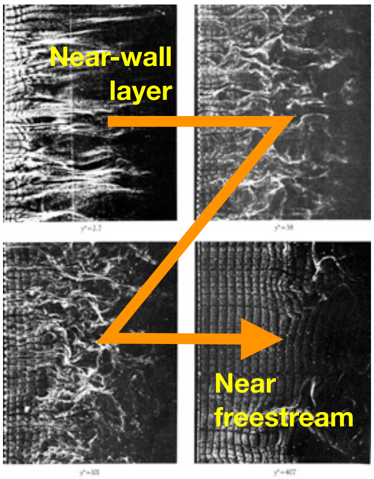
FLUID MECHANICS

Turbulent boundary layers

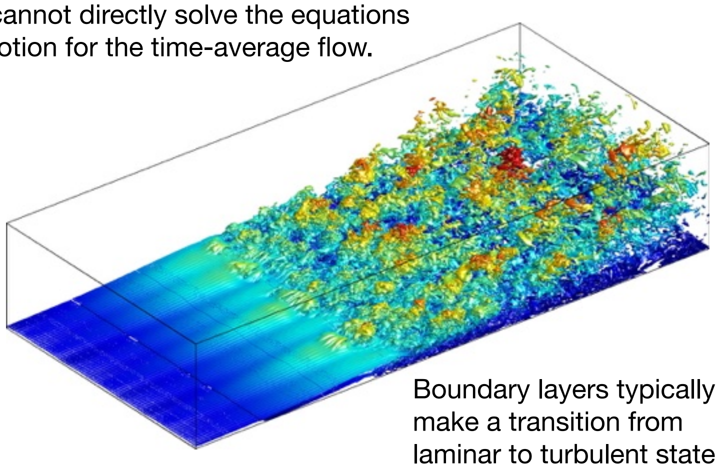


Structure varies with distance from wall.



Flow is disorderly and chaotic/random. Characterisation (e.g. of δ) requires averaging.

We cannot directly solve the equations of motion for the time-average flow.



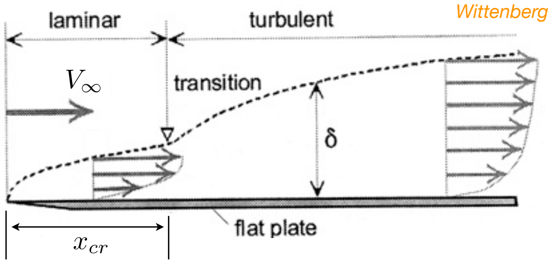
Boundary layers typically make a transition from laminar to turbulent state going downstream.

50

FLUID MECHANICS

Boundary layer flows, laminar and turbulent

Torenbeek & Wittenberg

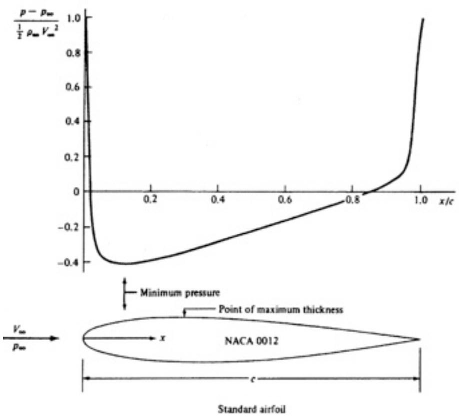


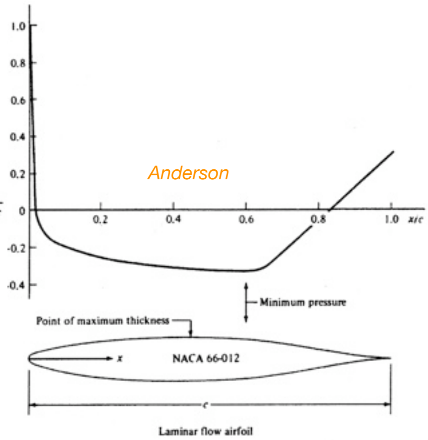
The process that transforms laminar to turbulent flow (and sometimes, the reverse) is called transition.

How far downstream transition occurs (say, at x_{cr}) depends largely on external flow fluctuations as well as V, ρ, μ. External flows that slow down in the streamwise direction promote transition at smaller values of x_{cr}) and vice versa.

Slowing external flow is associated (through Bernoulli's equation) with a external pressure that increases with x (i.e. dp/dx > 0). This is called an adverse pressure gradient, since we generally want to delay transition. Conversely, external flow that get faster with x is associated with an external pressure gradient that decreases with x (i.e. dp/dx < 0), and this situation is called a favourable pressure gradient).

We can change the external flow by varying the shape of the object on which the boundary layer grows, and hence influence somewhat if the boundary layer will be laminar (low drag) or turbulent (higher drag).





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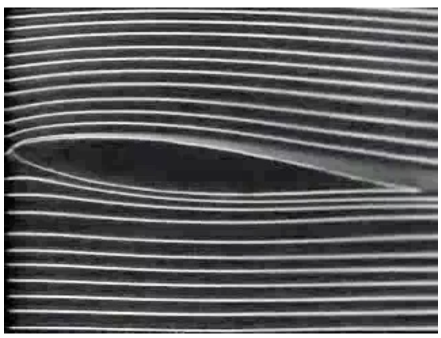
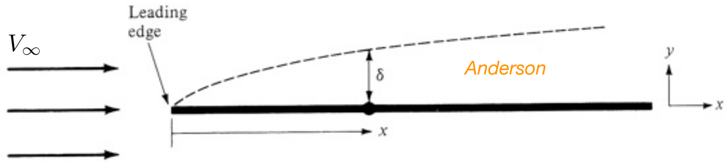


http://www.archive.org/details/BoundaryLayer

Boundary layer flows, laminar and turbulent

Laminar boundary layer on a flat plate

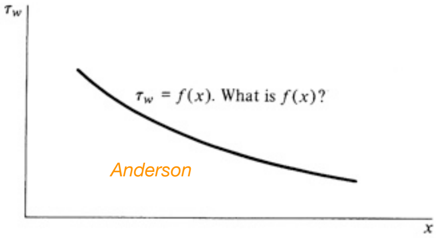
Two quantities of great interest for BL flows are the streamwise (x) distributions of BL thickness δ, and the shear stress at the wall, τ_w.



NB dimensionless

Laminar boundary layer theory gives $\delta_{\text{laminar}} = \dfrac{5.2x}{\sqrt{Re_x}}$ or $\dfrac{\delta_{\text{laminar}}}{x} = \dfrac{5.2}{\sqrt{Re_x}}$ (recall $Re_x = \rho V_\infty x/\mu$)

Hence $\delta_{\text{laminar}} \propto \sqrt{x}$ or $x \propto \delta^2$ – the shape of the BL boundary is parabolic, starting at the plate leading edge.



We already said the wall shear stress, τ_w, will fall with distance along the wall as the BL thickens.

It is typical to deal with wall shear stress in dimensionless form, too. We talk of the <u>local skin friction coefficient</u>

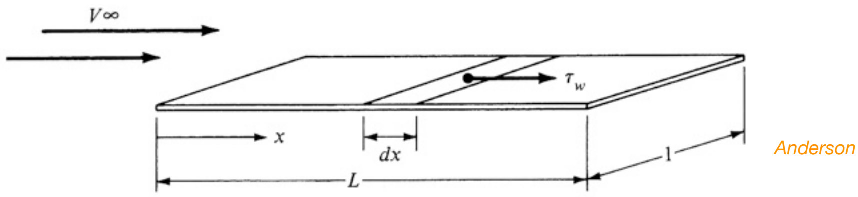
$$c_{f_x} = \frac{\tau_w}{\frac{1}{2}\rho V_\infty^2} = \frac{\tau_w}{q_\infty}$$

Laminar boundary layer theory gives $c_{f_x} = \dfrac{0.644}{\sqrt{Re_x}}$ (laminar) or $\tau_w = \dfrac{0.644 q_\infty}{\sqrt{Re_x}}$ i.e. $\tau_w \propto c_{f_x} \propto x^{-1/2}$

Boundary layer flows, laminar and turbulent

As well as the local skin friction coefficient, we are typically interested in the total drag per unit width, D_f', that the BL exerts on a flat plate of a given length, L. This is obtained by integration.

$$D_f' = \int_0^L \tau_w \, \mathrm{d}x = \int_0^L q_\infty \, c_{f_x} \, \mathrm{d}x$$



<u>For the laminar case</u> $c_{f_x} = \dfrac{0.644}{\sqrt{Re_x}}$ so that $D_f' = 0.644 q_\infty \displaystyle\int_0^L \frac{\mathrm{d}x}{\sqrt{Re_x}}$

$$D_f' = \frac{0.644 q_\infty}{\sqrt{\rho V_\infty/\mu}} \int_0^L \frac{\mathrm{d}x}{\sqrt{x}} = \frac{0.644 q_\infty}{\sqrt{\rho V_\infty/\mu}} 2\sqrt{L} = \frac{1.328 q_\infty L}{\sqrt{\rho V_\infty L/\mu}} = \frac{1.328 q_\infty L}{\sqrt{Re_L}}$$

where $Re_L = \dfrac{\rho V_\infty L}{\mu}$ is a Reynolds number defined for the overall plate length.

Define a total skin friction drag coefficient for the whole plate surface as

$$C_f = \frac{D_f'(1)}{q_\infty S} = \frac{D_f'}{q_\infty L} = \frac{1.328}{\sqrt{Re_L}} \quad \text{(laminar)}$$

(note Captial '*C*' here, as opposed to lower case '*c*' used for skin friction coefficient c_f.)

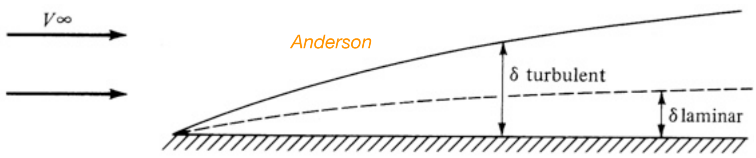
Note: although the total skin friction drag <u>coefficient</u> falls as L increases, the <u>actual drag force</u> increases.

FLUID MECHANICS

Boundary layer flows, laminar and turbulent

Turbulent boundary layer on a flat plate

Turbulent boundary layers both grow more quickly than do laminar ones, and produce more skin friction drag for the same thickness and free-stream speed.



Unlike laminar boundary layers, there are no pure theoretical results for turbulent boundary layers, and the relationships between variables have to be obtained experimentally.

$$\delta_{\text{turbulent}} \approx \frac{0.37x}{Re_x^{1/5}}$$

This means $\delta_{\text{turbulent}} \propto x^{4/5}$ compared to $\delta_{\text{laminar}} \propto x^{1/2}$ (faster growth)

And

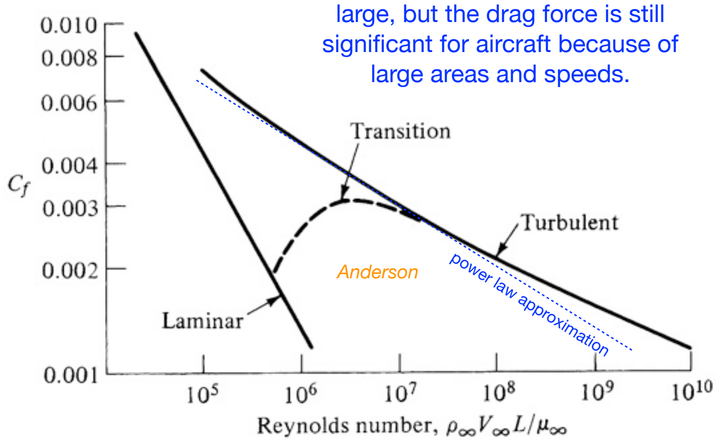
$$c_{f_x} \approx \frac{0.0592}{Re_x^{1/5}} \quad \text{(turbulent)}$$

$$C_f \approx \frac{0.074}{Re_L^{1/5}} \quad \text{(turbulent)}$$

Note: these power laws are only approximations.

Interestingly, typical overall plate length Reynolds numbers for a switch between one regime and the other are broadly similar to wing chord Reynolds numbers for a range of aircraft. This implies the switch is likely to be an issue in many cases of practical use.

These drag coefficients are not large, but the drag force is still significant for aircraft because of large areas and speeds.



FLUID MECHANICS

Example: supertanker power consumption

Assuming that most of the drag on a supertanker is produced by the immersed BL, estimate the power required to drive a tanker which is 300m long and 50m wide at a speed of 20km/hr.

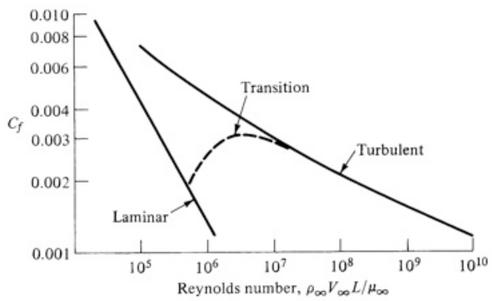
Approximate the immersed hull as a rectangular flat plate.

Sea water: ρ = 1025 kg/m³, μ = 1 × 10⁻³ Pa.s.

$$D = \tfrac{1}{2}\rho V^2 S C_f$$

$$V = 20/3.6 \text{ m/s} = 5.55 \text{ m/s}$$

$$S = L \times b = 300 \times 50 \text{ m}^2 = 15 \times 10^3 \text{ m}^2$$



$$Re_L = \frac{\rho V L}{\mu} = \frac{1025 \times 5.55 \times 300}{1 \times 10^{-3}} = 1.71 \times 10^9$$

We can reasonably assume the entire BL is turbulent.

$$C_f \approx \frac{0.074}{Re_L^{1/5}} \quad \text{(turbulent)}$$

$$C_f = \frac{0.074}{(1.71 \times 10^9)^{0.2}} = 1.05 \times 10^{-3}$$

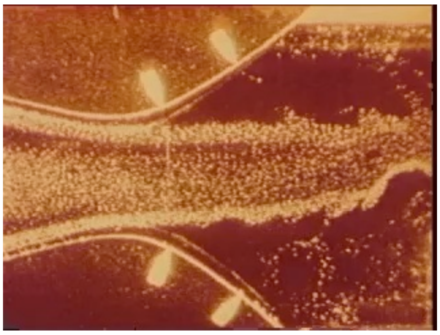
$$D = \tfrac{1}{2}\rho V^2 S C_f$$

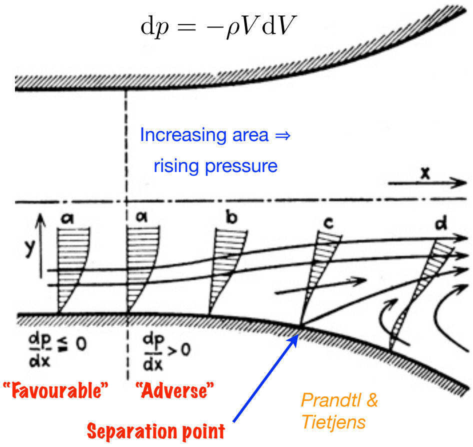
$$= 0.5 \times 1025 \times 5.55^2 \times 15 \times 10^3 \times 1.05 \times 10^{-3} \text{ N}$$

$$= 248.6 \text{ kN}$$

$$P = DV = 248.6 \times 10^3 \times 5.55 \text{ W} = \underline{1.38 \text{ MW}}$$

Flow separation





Flow separation occurs where boundary layer flows are unable to conform to the shape of an object, and the flow moving in the downstream direction 'separates' from the wall at a location called a separation point (in 2D) or along a separation line (in 3D). Downstream of a separation point, there is slow counter-current flow near the wall.

Flow separation is a major problem in aeronautics, because it almost invariably leads to an increase in drag. Over an otherwise streamlined object, the freestream flow is changed, influencing pressure distributions, often to the point that drag caused by pressure differentials between the front and back of a body causes 'pressure drag' that can greatly exceed the viscous skin friction drag.

Flow separation is very often associated with an **adverse pressure gradient** – pressure increasing in the streamwise direction.

In this situation, Euler's equation suggests that the velocity will decrease. **For fluid near the wall, velocities are already small, and the imposed pressure gradient can eventually bring fluid there to rest, and indeed to reverse direction – leading to separation.**

The whole process is a delicate relationship between momentum, pressure gradients, and viscous stress.

Importantly, turbulent boundary layers are better at resisting flow separation, since fluid near the wall tends to have more momentum than would be the case if the flow were laminar. Perhaps surprisingly this may lead to a lower overall drag even though the boundary layer component can be larger.

Drag: viscous skin friction vs pressure drag

We note that drag can be created both by viscous skin friction (skin friction drag) and by pressure differences front-to-rear (pressure drag). For 2D bodies, BOTH these effects need viscosity to produce them – neither would be present in an ideal frictionless flow.

Large pressure drag is typically associated with flow separation, although in fact it can also be present in the absence of separation.

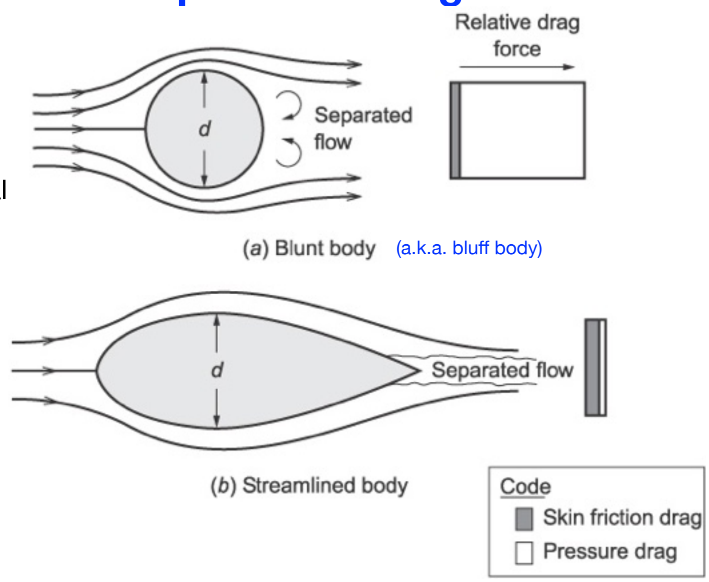
$$D = D_f + D_p$$

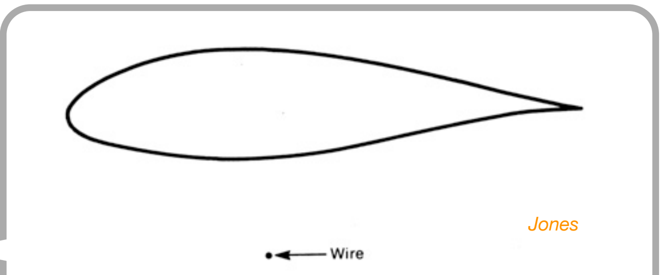
Total drag due to viscous effects | Drag due to skin friction | Drag due to separation (pressure drag)

The relative contributions of skin friction and pressure drag depends on whether the flow has a large or small amount of flow separation.

Note that pressure drag is by far the dominant component when there is a large amount of separation (for flow around a bluff body).

Because of this a bluff body with small cross-section (or projected area in the direction normal to the free-stream flow) can produce the same total amount of drag as a streamlined body of much larger projected area.

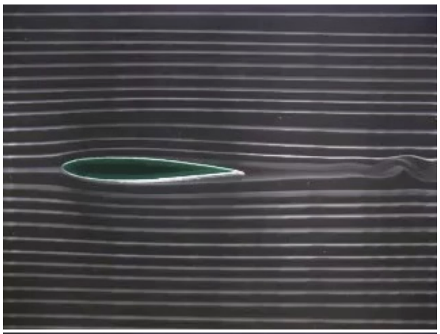




1.2. NACA 64-421 airfoil compared with a circular wire having the same drag.

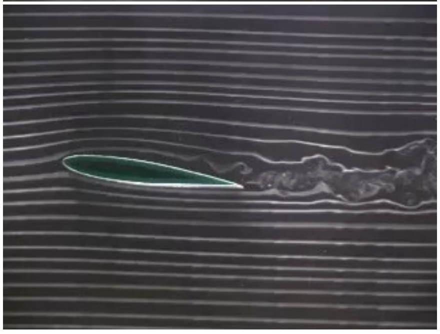
Flow separation and stall

Perhaps the single most significant flow-separation phenomenon in aerodynamics is the separation of flow over the top of an airfoil as angle of attack α is increased.

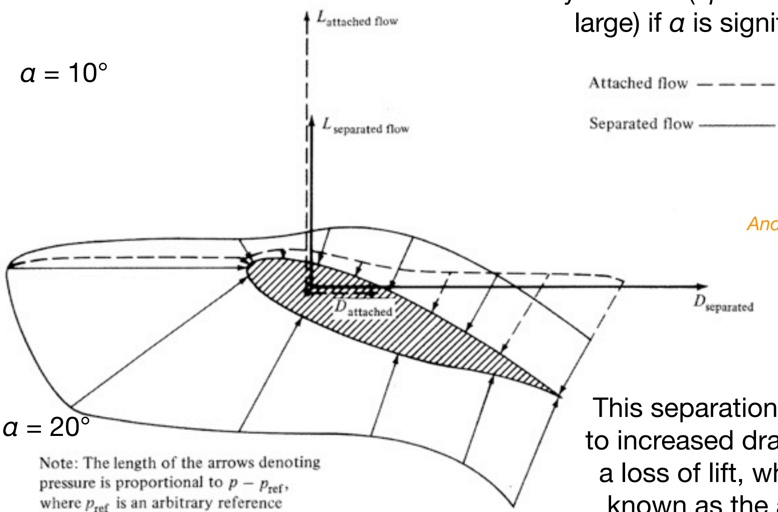


$\alpha = 0°$

In this case all the action leading to flow separation typically occurs around the upper leading edge of the airfoil, where the pressure gradient is very adverse (dp/dx is very large) if α is significant.



$\alpha = 10°$



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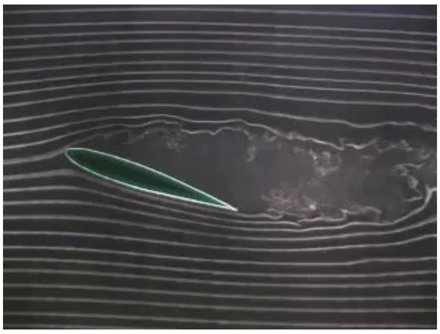

$\alpha = 20°$

Note: The length of the arrows denoting pressure is proportional to $p - p_{\text{ref}}$, where p_{ref} is an arbitrary reference pressure slightly less than the minimum pressure on the airfoil.

This separation leads to increased drag and a loss of lift, which is known as the airfoil/wing/aircraft stalling.