1

Aircraft performance requirements and design

Recommended reading:

Torenbeek & Wittenberg: Chapter 6
Nicolai & Carichner: Chapter 3
Tonerbeek: Chapter 5
Brandt et al.: Chapter 5

2

Performance requirements — 1

Civil-aviation-type performance requirements are typically obtained from design range and payload and standardized (international) safety regulations (which cover much more than aircraft performance).

1. Takeoff: distance required to take-off at a safe flying speed and clear a barrier (50ft/15m), with landing gear extended, flaps in takeoff position, aircraft at maximum takeoff mass (MTOM)
2. Second-segment climb: a minimum gradient of climb, with landing gear retracted, flaps in takeoff configuration, aircraft at MTOM and with one engine inoperative (OEI), up to 400ft.
3. Cruise-climb: at initial cruise height, aircraft must be able to climb another 500ft with a climb rate of at least 300ft/min (1.5m/s).
4. Cruise: carry a payload over a distance (with minimum fuel use)
5. Missed approach: in landing configuration and at approach speed, climb at a minimum gradient.
6. Diversion: reserve fuel for diversion to another airport.
7. Landing: distance required to clear a 50ft barrier and brake to rest without thrust reversers, starting from a safe approach speed, landing configuration.

Military-aviation-type performance requirements are broadly similar but for combat aircraft, manoeuvre/ air-superiority/interception requirements may be added (and are often more demanding), e.g.

1. Minimum cruise and/or short-term dash speed, at various altitudes.
2. Minimum service ceiling.
3. Minimum acceleration at various altitudes.
4. Minimum time to reach a given altitude/speed (energy state) from SL.
5. Minimum instantaneous turn rate/radius at various altitudes.
6. Minimum sustained turn rate/radius at various altitudes.

Combat aircraft performance requirements are often set with regard to opposing threats/aircraft.

3

Performance requirements — 2

Extract from the
Australian Civil Aviation Safety Regulations

Summary:

Australian civil aviation safety certification regulations (CASRs) basically follow US Federal Airworthiness Regulations (**FAR**s).

Small civil aircraft	FAR 23
Larger civil aircraft	FAR 25
Small civil rotorcraft	FAR 27
Larger civil rotorcraft	FAR 29

International/European regulations are called Joint Airworthiness Regulations (**JAR**s), have same numbering as FARs.

Type of Aircraft	Date accepted for operational use by the Armed Force	Regulations that apply [1]
Small reciprocating-engine powered aeroplanes	Before May 16, 1956	Civil Air Regulations Part 3, as effective May 15 1956
	After May 15, 1956	Civil Air Regulations Part 3, or FARs Part 23, or CASR Part 23
Small turbine-engine powered aeroplanes	Before Oct. 2, 1959	Civil Air Regulations Part 3, as effective Oct.1 1959
	After Oct. 1, 1959	Civil Air Regulations Part 3, or FARs Part 23, or CASR Part 23
Commuter category aeroplanes	After Feb. 17, 1987	FARs Part 23, as effective Feb 17, 1987, or CASR Part 23
Large reciprocating-engine powered aeroplanes	Before Aug. 26, 1955	Civil Air Regulations Part 4b, as effective Aug. 25, 1955
	After Aug. 25, 1955	Civil Air Regulations Part 4b, or FARs Part 25, or CASR Part 25
Large turbine engine-powered aeroplanes	Before Oct. 2, 1959	Civil Air Regulations Part 4b, as effective Oct. 1, 1959
	After Oct. 1, 1959	Civil Air Regulations Part 4b, or FARs Part 25, or CASR Part 25
Rotorcraft with a maximum certificated take-off weight of:		
2,722 kg or less	Before Oct. 2, 1959	Civil Air Regulations Part 6, as effective Oct. 1, 1959
	After Oct. 1, 1959	Civil Air Regulations Part 6, or FARs Part 27, or CASR Part 27
Over 2,722 kg	Before Oct. 2, 1959	Civil Air Regulations Part 7, as effective Oct. 1, 1959
	After Oct. 1, 1959	Civil Air Regulations Part 7, or FARs Part 29, or CASR Part 29.

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Certification performance tests

List of performance ground and flight tests required for a certificate of airworthiness

1. Calibration of pitot-static system
   - with a towed static tube as a reference at low speed
   - with a towed static cone as a reference at high speed
   - calibration during ground runs.

2. Calibration of angle-of-attack and outside-air temperature sensors

3. Determination of stall speed and stall characteristics

4. Determination of minimum control speeds with one engine inoperative (V_{MCA} and V_{MCG})

5. Determination of the take-off performance
   - minimum unstick speed (V_{MU})
   - rotation speed (V_R)
   - lift-off speed (V_{LOF})
   - recognition time engine failure
   - runway length over 35ft obstacle
   - accelerate-stop length
   - take-off performance with an early rotation before the established V_R

   These characteristics have to be determined with
   - all engines operative
   - one engine inoperative (for two-engined aircraft)
   - every flap setting for which the aircraft should be able to take-off

6. Landing performance over a 50 ft obstacle (for all landing flap settings)

7. Buffet-boundary at high Mach numbers

8. Climb performance in
   - take-off configuration (undercarriage up and down)
   - cruise configuration
   - approach configuration
   - landing configuration

9. Engine calibration in an altitude test facility

10. Determination of inlet and exhaust efficiency

Note:
Although the following subjects are in general not part of the basic certification procedure, with regard to field performance tests are also performed on
- wet runways
- runways with wet snow
- the effect of failure of one of the braking devices of the aircraft

Figure 44.18 - List of performance-related ground and flight tests required for a certificate of airworthiness of a large civil transport aircraft

Obert

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Design for performance

In most of the performance analysis to follow, we don't usually explicitly relate aircraft weight back to MTOW W_0, or explicitly relate the propulsion system output back to maximum rated values, i.e. T_0 or $P_{s,0}$.

However, when it comes to inverting the performance analysis to use for design, we will need to do these conversion. So it is useful to remember the implicit relationships

Weight: $W = \beta W_0$ and/or $\frac{W}{S} = \beta \frac{W_0}{S}$

Available thrust (jet): $T_A = \alpha(V, h, \text{throttle setting}) T_0$

Available power (prop): $P_A = \eta_{\text{pr}} \alpha(V, h, \text{throttle setting}) P_{s,0}$

It is also useful to note that we generally treat the aircraft as an equivalent point mass (located at the aircraft CG) and ignore rotational inertias and motions, as well as aircraft flexibility. These things are of course important, but usually are not considered in aircraft initial design.

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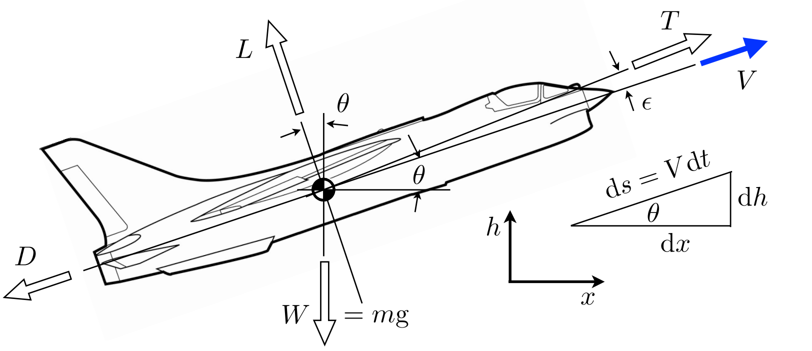
Aircraft equations of motion



First consider flight where there is no bank angle, and all forces and motion occur in the aircraft plane of symmetry.

The aircraft is assumed to be trimmed so that moments sum to zero.

From flight path geometry we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = V\cos\theta$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \dot{h} = V\sin\theta$$

Newton's second law is $$\sum \boldsymbol{F} = \frac{\mathrm{d}(m\boldsymbol{V})}{\mathrm{d}t} = m\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} + \boldsymbol{V}\frac{\mathrm{d}m}{\mathrm{d}t} = m\dot{\boldsymbol{V}} + \boldsymbol{V}\dot{m} \approx m\dot{\boldsymbol{V}}$$

Note that it is usual in aircraft performance dynamics to ignore the time rate of change of mass, except when computing fuel consumption. This assumption may be inadequate when fuel burn rates are high, e.g. for performance analysis of missiles.

Now we consider components tangential and normal to the flight path, leading to

Tangential $$m\frac{\mathrm{d}V}{\mathrm{d}t} = T\cos\epsilon - D - mg\sin\theta = m\dot{V}$$

Normal $$mV\frac{\mathrm{d}\theta}{\mathrm{d}t} = T\sin\epsilon + L - mg\cos\theta = mV\dot{\theta} \equiv m\frac{V^2}{R_V}$$ where R_V is flight path radius of curvature.

Rearrange:

(Divide tangential equation through by *mg* = *W*.) $$\frac{\dot{V}}{g} = \frac{T\cos\epsilon - D}{W} - \sin\theta$$

(Divide normal equation through by *mgV*.) $$\frac{\dot{\theta}}{g} = \frac{T\sin\epsilon + L}{VW} - \frac{\cos\theta}{V}$$

Aircraft equations of motion

Since the thrust and lift are functions of altitude and speed, the drag is additionally a function of lift, and the fuel burn rate depends on thrust, we have a set of five ODEs to consider, four of which are coupled:

Note coupling of terms between equations: in general, cannot solve them independently

Integrate to get range $\dot{x} = V\cos\theta$ (not directly coupled to the remaining four)

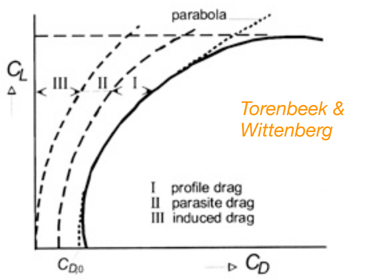
Integrate to get *h* $\dot{h} = V\sin\theta$

Integrate to get *W* $\dot{W} = -\mathrm{g}c_t(h,V)T(h,V)$

Integrate to get *V* $\dfrac{\dot{V}}{\mathrm{g}} = \dfrac{T(h,V)\cos\epsilon - D(h,V,L)}{W} - \sin\theta$

Integrate to get *θ* $\dfrac{\dot{\theta}}{\mathrm{g}} = \dfrac{T(h,V)\sin\epsilon + L(h,V)}{VW} - \dfrac{\cos\theta}{V}$

Drag polar $C_D = C_{D,0} + KC_L^2$



The standard approach for most performance analysis is to obtain decoupling by assuming that θ=const., and base everything around the 4th equation in the set. i.e. the one containing the drag polar. (Also, since ϵ is typically small or zero, cosϵ→1 and sinϵ→ϵ.) Thus, starting with the 4th equation,

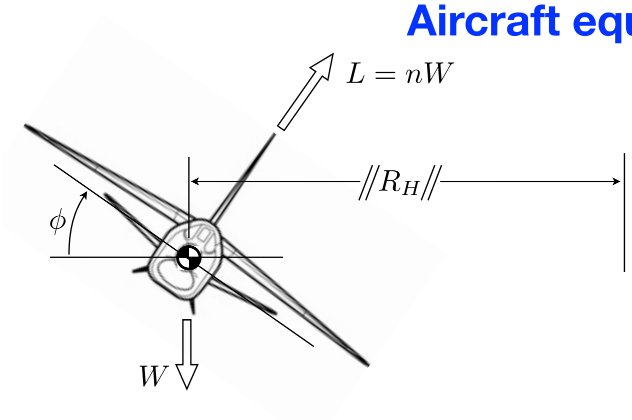
$$\frac{(T-D)V}{W} = V\sin\theta + V\frac{\dot{V}}{\mathrm{g}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(h + \frac{V^2}{2\mathrm{g}}\right) = \frac{\mathrm{d}e}{\mathrm{d}t}$$

Fundamental Performance Equation

where *e*, the sum of potential and kinetic energies per unit weight, is called the aircraft's specific energy or energy height. The equation is often called the Fundamental Performance Equation and is the basis for most performance analysis (and design for performance). The rate of change of KE is often ignored.

The term (*T*–*D*)/*W* is called the specific excess thrust and (*T*–*D*)*V*/*W* is called the specific excess power, i.e. the amount of thrust/power per unit weight available to increase the aircraft's altitude or speed, or both.

Aircraft equations of motion



In manoeuvres, we seek to change the heading of the aircraft, and a commonly-used basis for analysis is to assume that the aircraft executes a coordinated horizontal banking turn of constant radius R_H and at constant speed *V*.

For now we will also assume that ϵ=0, and hence that the thrust makes no direct contribution to turning. It is simple enough to restore it if required.

Note the use of the load factor *n* to describe the lift. This corresponds to the increased wing bending load.

Horizontal $L\sin\phi = m\dfrac{V^2}{R_H} = mV\omega$

Vertical $L\cos\phi = nW\cos\phi = W,\quad n\cos\phi = 1,\quad \cos\phi = \dfrac{1}{n},\quad \phi = \cos^{-1}\left(\dfrac{1}{n}\right)$

$$\cos^2\phi + \sin^2\phi = 1,\quad \sin^2\phi + \frac{1}{n^2} = 1,\quad \sin^2\phi = 1 - \frac{1}{n^2} = \frac{n^2-1}{n^2},\quad \sin\phi = \frac{\sqrt{n^2-1}}{n}$$

Substitute into horizontal component equation:

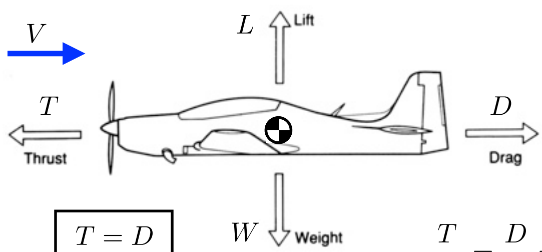
$$\frac{nW\sqrt{n^2-1}}{n} = mV\omega,\quad m\mathrm{g}\sqrt{n^2-1} = mV\omega,\quad \omega = \frac{\mathrm{g}\sqrt{n^2-1}}{V}$$

Equivalently: $R_H = \dfrac{V^2}{\mathrm{g}\sqrt{n^2-1}}$

Level unaccelerated flight

Thrust and speed — 1

Recall



$$T = D \quad W = L$$

$$L = \tfrac{1}{2}\rho V^2 S C_L = W \longrightarrow C_L = \frac{2}{\rho V^2}\frac{W}{S} = \frac{2}{\rho_0 V_e^2}\frac{W}{S}$$

$$D = \tfrac{1}{2}\rho V^2 S C_D = T \qquad V = \sqrt{\frac{2}{\rho}\frac{W}{S}\frac{1}{C_L}}$$

$$\frac{T}{W} = \frac{D}{L} = \frac{C_D}{C_L} = \frac{1}{C_L/C_D} \qquad \left(\frac{T}{W}\right)_{\min} = \left(\frac{D}{L}\right)_{\min} = \frac{1}{(C_L/C_D)_{\max}}$$

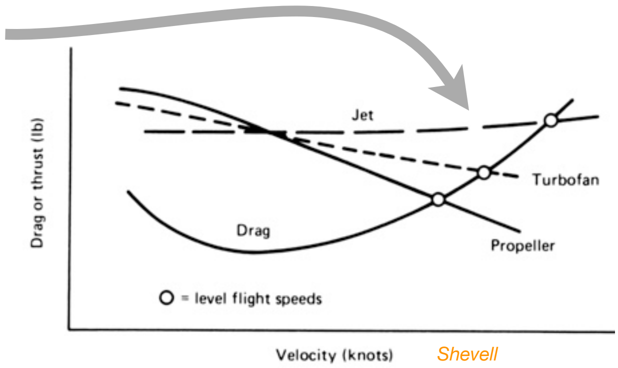
Minimum drag, or thrust, for given weight (T/W) occurs at minimum C_D/C_L, i.e. maximum C_L/C_D.

Now if $C_D/C_L = C_{D,0}/C_L + KC_L$, this has a minimum where $C_{D,0}/C_L^* = KC_L^*$ i.e.

$C_L^* = \sqrt{C_{D,0}/K}$ and $(C_D/C_L)^* = 2\sqrt{C_{D,0}K}$ or $(C_L/C_D)^* = 1/\sqrt{4C_{D,0}K}$

For the case that the powerplant's thrust is approximately independent of speed (i.e. a jet, at cruise), a number of questions arise:

1. What is the minimum thrust required to fly?
2. What is the corresponding speed (either TAS or EAS)?
3. For a larger thrust, what is the speed?
4. What is the maximum altitude the aircraft can attain?
5. What is the speed/altitude envelope of the aircraft?



Thrust and speed — 2

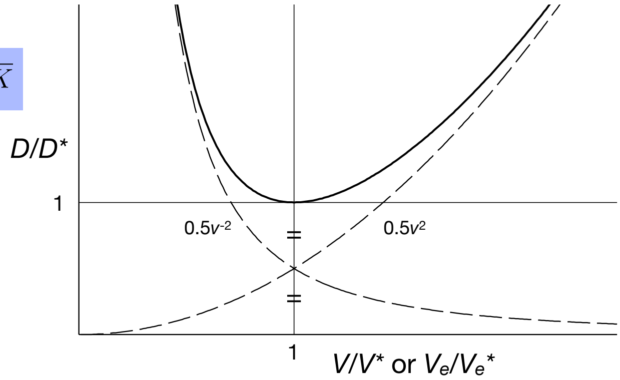
1. What is the minimum thrust required to fly?

$$\left(\frac{T}{W}\right)_{\min} = \left(\frac{D}{L}\right)^* = 2\sqrt{C_{D,0}K}, \quad T_{\min} = 2W\sqrt{C_{D,0}K}$$

Note that this does not depend on altitude.
Reducing W, $C_{D,0}$ or K reduces $T_{\min}$.

We recall that for this condition, the zero-lift drag equals the induced drag:

$$C_D^* = C_{D,0} + KC_L^{*2} = C_{D,0} + KC_{D,0}/K = 2C_{D,0}$$



Induced and parasitic drag are equal at the minimum.

2. What is the minimum-thrust speed?

EAS

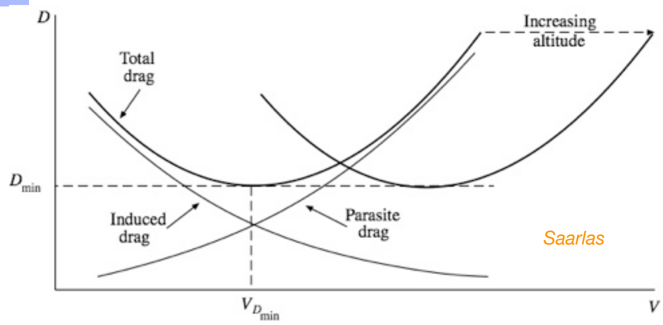
$$V_e^* = \left(\frac{2}{\rho_0}\frac{W}{S}\right)^{1/2}\left(\frac{1}{C_L^*}\right)^{1/2} = \left(\frac{2}{\rho_0}\frac{W}{S}\right)^{1/2}\left(\frac{K}{C_{D,0}}\right)^{1/4}$$

TAS

$$V^* = \left(\frac{2}{\rho}\frac{W}{S}\right)^{1/2}\left(\frac{1}{C_L^*}\right)^{1/2} = \frac{V_e^*}{\sqrt{\sigma}}$$

We see that minimum-drag (minimum-thrust) TAS increases with altitude, while the drag stays fixed.

Also, increasing W/S or reducing $C_L{}^*$ raises V^*.



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Thrust and speed — 3

3. What is the speed for a thrust $T_A > T_{\min}$?

We see that there are two possible solutions. →

Caveat: the aircraft may stall at an airspeed above the lesser of the two.

$$\frac{T_A}{W} = \frac{D}{W} = \frac{D}{L} = \left(\frac{C_D}{C_L}\right)^* \frac{u^2 + u^{-2}}{2} = \left(\frac{C_D}{C_L}\right)^* \frac{u^4+1}{2u^2}$$

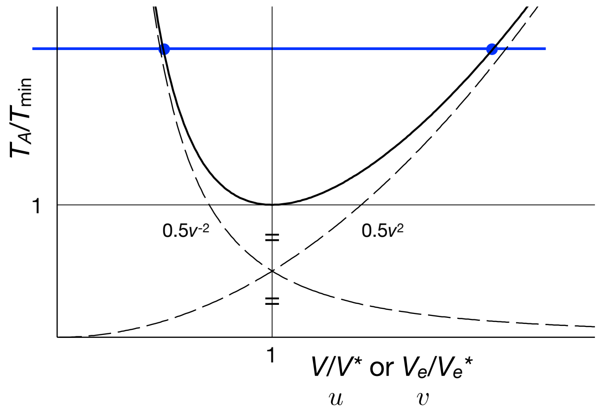
Rearrange:

$$u^4 - 2\frac{T_A}{W}\left(\frac{C_L}{C_D}\right)^* u^2 + 1 = 0$$

Solve quadratic in u^2:

$$u^2 = \frac{T_A}{W}\left(\frac{C_L}{C_D}\right)^* \pm \left(\left[\frac{T_A}{W}\left(\frac{C_L}{C_D}\right)^*\right]^2 - 1\right)^{1/2}$$

$$u = \left\{\frac{T_A}{W}\left(\frac{C_L}{C_D}\right)^* \pm \left(\left[\frac{T_A}{W}\left(\frac{C_L}{C_D}\right)^*\right]^2 - 1\right)^{1/2}\right\}^{1/2}$$



One can show that $u_1 u_2 = 1$

TAS $$V = V^* u = \left(\frac{2}{\rho}\frac{W}{S}\frac{1}{C_L^*}\right)^{1/2} u$$

EAS $$V_e = V_e^* u = \sqrt{\sigma} V^* u$$

Alternatively (check!)

$$V = \left\{\frac{(T_A/W)(W/S) \pm (W/S)\left([T_A/W]^2 - 4C_{D,0}K\right)^{1/2}}{\rho C_{D,0}}\right\}^{1/2}$$

Variation of T_R with altitude, configuration, weight

We have already seen that for a given aircraft, the drag (or thrust required to fly, T_R), does not vary with altitude (i.e. σ), though the TAS for a given drag does.

However, increasing W (i.e. W/S) at fixed altitude increases the minimum thrust and the speed at which that occurs.

And changing aircraft configuration (drag polar) at fixed altitude and weight also changes things.

All these effects are simple consequences of

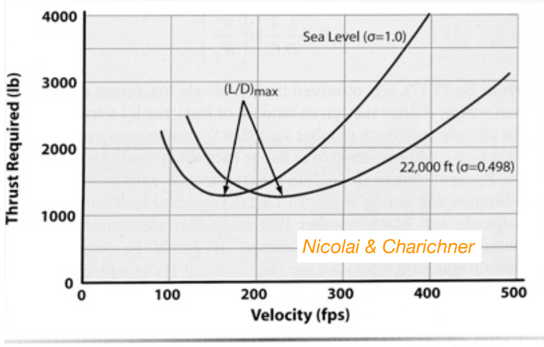


Figure 3.6 Effect on T_R of changing aircraft altitude ($\sigma = \rho/\rho SL$).

$$T = \frac{W}{C_L/C_D} = \frac{W}{C_L/(C_{D,0} + KC_L^2)} \quad \text{and} \quad W = L = \tfrac{1}{2}\rho V^2 S C_L$$

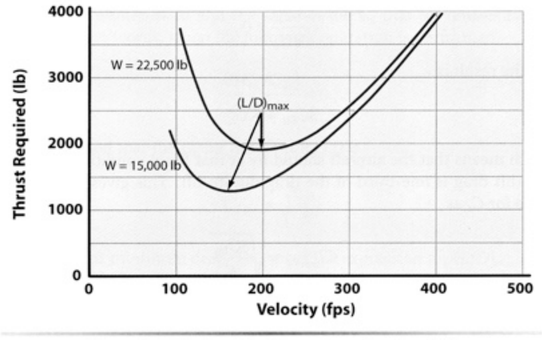


Figure 3.4 Effect on T_R of changing aircraft weight.

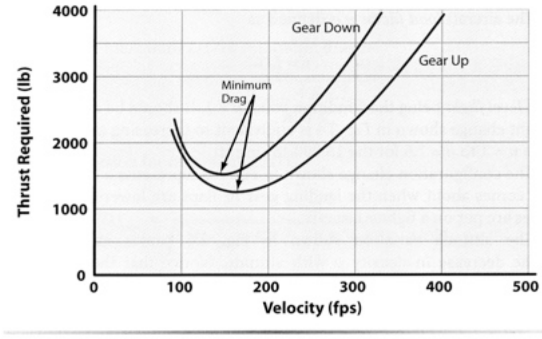


Figure 3.5 Effect on T_R of changing aircraft configuration.

Thrust and speed — 4

4. What is the maximum altitude the aircraft can attain?

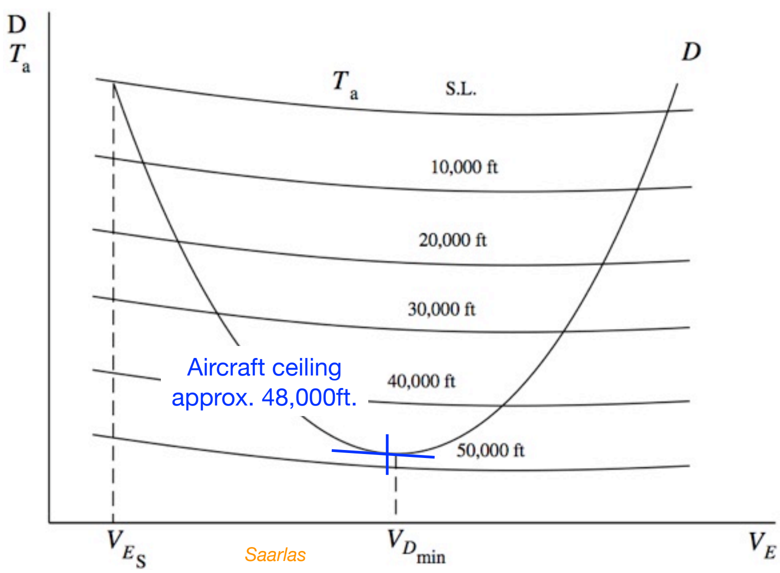
While the thrust required does not change with altitude, the thrust available falls.

It is most convenient to plot the drag curve with V_e as abscissa, since it is the same at all altitudes.

Over this we plot the available powerplant thrust as a function of V_e and h.

The altitude at which the available thrust falls below the minimum drag is the aircraft's ceiling.

The approach here is graphical but if we can describe/curve fit the propulsion system's thrust lapse with speed and altitude then we can solve the problem numerically. If we assume thrust is independent of speed and just depends on altitude, the solution becomes quite simple.



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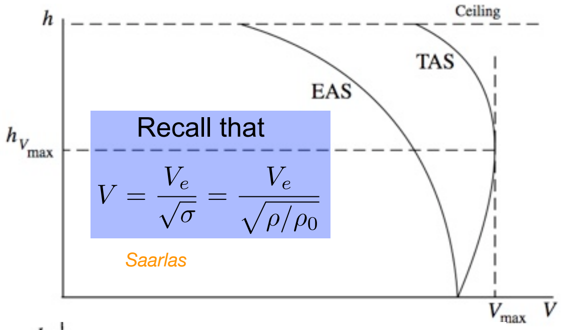
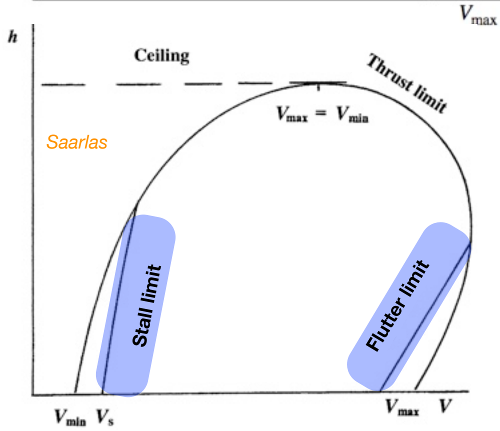
Thrust and speed — 5

5. What is the speed/altitude envelope of the aircraft?

We just found the minimum and maximum EASs at each altitude. Next we convert to TAS.

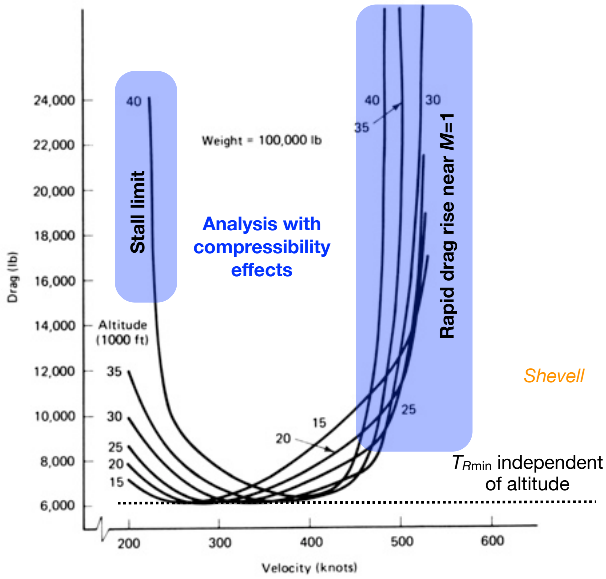
While maximum EAS falls with altitude, TAS will initially increase, then decrease.





Typically there will be additional limits imposed by C_{Lmax}/stall at low speeds and buffet/flutter effects at high speeds.

The discussion above does not allow for compressibility. Wave drag may also substantially reduce maximum TAS.



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Thrust and speed — 6

6. Speed limitations

The minimum speed achievable is limited by stall, a nonlinear effect not accounted for in our two-parameter drag polar.

$$V_{\text{stall}} = \sqrt{\frac{2}{\rho}\frac{W}{S}\frac{1}{C_{L\,\max}}}$$

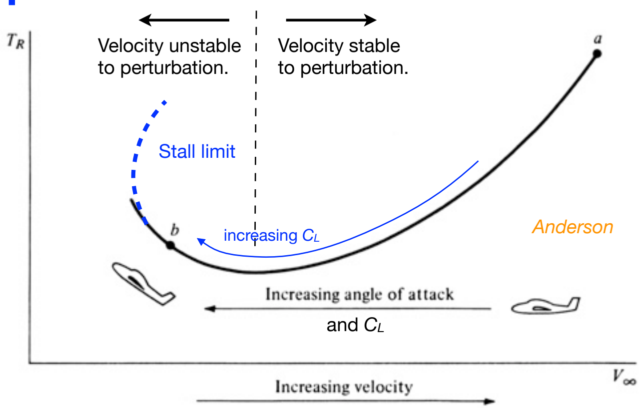
$$V_{e,\text{stall}} = \sqrt{\frac{2}{\rho_0}\frac{W}{S}\frac{1}{C_{L\,\max}}} = \text{const.}$$

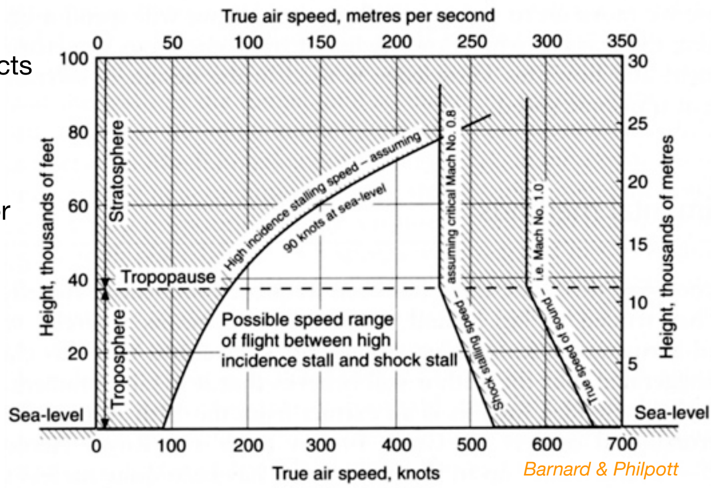
Landing approach speeds are typically given as some factor (e.g. 1.3) above V_{stall}.

Maximum speed may be limited by aeroelastic effects (maximum IAS) or by shock buffet (maximum Mach number).

Stall and shock buffet speeds converge as altitude increases. This presents an operational difficulty for high-altitude subsonic aircraft.

Finally, we note that of course we can achieve lower speeds than the maximum predicted values simply by reducing the throttle setting. However, most aircraft engines become rather inefficient if throttled back significantly below their design values.





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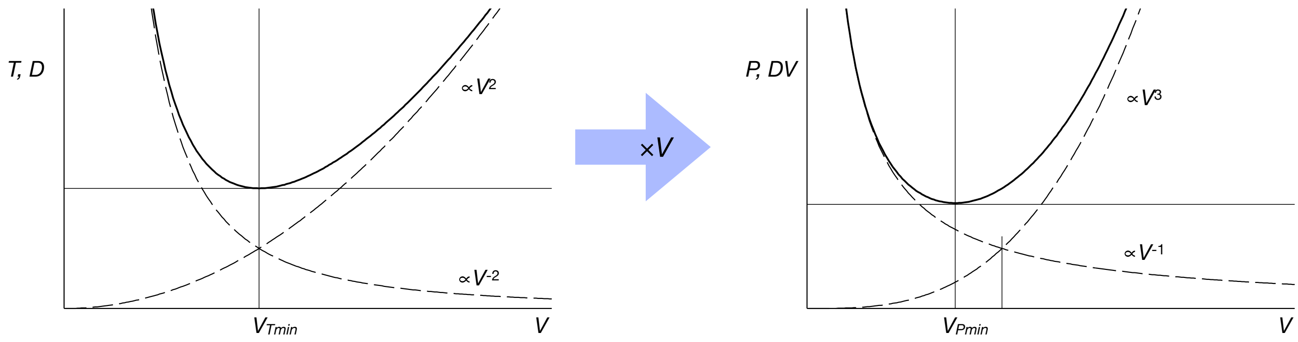
Power and speed — 1

If engine performance is better characterised by available power than thrust and e.g. we are interested in minimum power required to fly, the analysis is similar but the outcomes somewhat different.

The aircraft thrust-speed curve is converted to a power-speed curve by multiplying through by velocity, since $P=TV=DV$.



<u>We see that the minimum-power speed is lower than the minimum-thrust speed.</u>

Following our previous approach for thrust-related performance we seek answers to the questions:

1. What is the minimum <u>power</u> required to fly?
2. What is the corresponding speed (either TAS or EAS)?
3. For a larger power, what is the speed?
4. What is the maximum altitude the aircraft can attain?
5. What is the speed/altitude envelope of the aircraft?
   (Essentially an adaptation of the method used for thrust-related case.)

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Power and speed — 2

<u>1. What is the minimum power required to fly?</u> and <u>2. What is the corresponding speed?</u>

$$P = DV = \tfrac{1}{2}\rho V^2 S C_D V; \qquad V^3 = \left(\frac{2}{\rho}\frac{W}{S}\frac{1}{C_L}\right)^{3/2}$$

$$= \frac{2}{\rho} S \frac{C_D}{C_L^{3/2}} \left(\frac{2}{\rho}\frac{W}{S}\right)^{3/2}$$

$$= \left(\frac{2}{\rho}\frac{W}{S}\right)^{1/2} W \frac{C_D}{C_L^{3/2}} \quad \Rightarrow \quad \frac{P}{W} = \left(\frac{2}{\rho}\frac{W}{S}\right)^{1/2} \frac{C_D}{C_L^{3/2}}$$

<u>Minimise power requirement by flying at $(C_L^{3/2}/C_D)_{max}$.</u>

Now $\dfrac{C_L^{3/2}}{C_D} = C_L^{1/2}\dfrac{C_L}{C_D} = C_L^{1/2}\left(\dfrac{C_L}{C_D}\right)^* \dfrac{2}{u^2+u^{-2}} = C_L^{1/2}\left(\dfrac{C_L}{C_D}\right)^* \dfrac{2u^2}{u^4+1}$ where $\left(\dfrac{C_L}{C_D}\right)^* = \dfrac{1}{2\sqrt{C_{D,0}K}}$

and $u = \dfrac{V}{V^*}; \quad V^* = \left(\dfrac{2}{\rho}\dfrac{W}{S}\right)^{1/2}\dfrac{1}{(C_L^*)^{1/2}} = \left(\dfrac{2}{\rho}\dfrac{W}{S}\right)^{1/2}\left(\dfrac{K}{C_{D,0}}\right)^{1/4}$ so $(C_L)^{1/2} = \dfrac{(C_L^*)^{1/2}}{u}$

$$\therefore \quad \frac{P}{W} = \left(\frac{2}{\rho}\frac{W}{S}\right)^{1/2}\frac{1}{(C_L^*)^{1/2}}\left(\frac{C_D}{C_L}\right)^*\frac{u^4+1}{2u} = \left(\frac{2}{\rho}\frac{W}{S}\right)^{1/2} 2\left(C_{D,0}K^3\right)^{1/4}\frac{u^4+1}{2u}$$

Constant with speed

Minimised where $u=(1/3)^{1/4}=0.760$.

I.e. P/W is least when $C_L^{3/2}/C_D$ is maximized, <u>at speed $V=0.760\times V^*$.</u>

(At this speed, induced drag is three times larger than the zero-lift drag, and $C_L/C_D= (3/4)^{1/2}\times(C_L/C_D)_{max}$.)

Substituting in $u=(1/3)^{1/4}$, we find that

$$\left(\frac{P}{W}\right)_{min} = \left(\frac{256}{27}\right)^{1/4}\left(\frac{2}{\rho}\frac{W}{S}\right)^{1/2}\left(C_{D,0}K^3\right)^{1/4} = 1.755\left(\frac{2}{\rho}\frac{W}{S}\right)^{1/2}\left(C_{D,0}K^3\right)^{1/4}$$

1. Best to fly at minimum altitude, largest ρ. 2. More important to reduce K than $C_{D,0}$.

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Power and speed — 3

The minimum power required (P_r) to fly increases with altitude, owing to the $\rho^{-1/2}$ term. The corresponding TAS also increases.

<u>3. For a power larger than the minimum required to fly, what is the speed?</u>

Even in the case that power is constant with speed, we have to carry out a numerical/graphical solution of

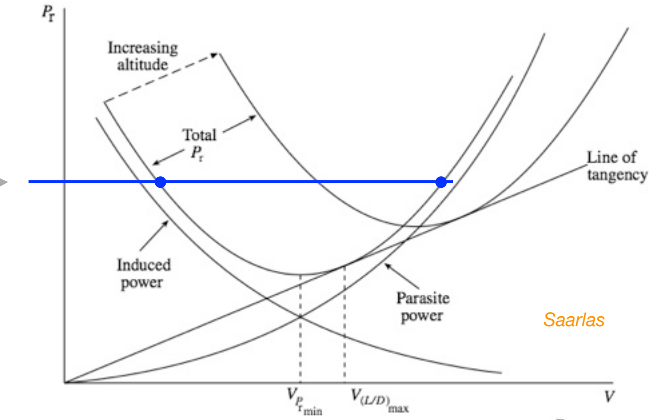
$$\frac{P}{W} = \left(\frac{2}{\rho}\frac{W}{S}\right)^{1/2} \frac{1}{(C_L^*)^{1/2}} \left(\frac{C_D}{C_L}\right)^* \frac{u^4+1}{2u}$$

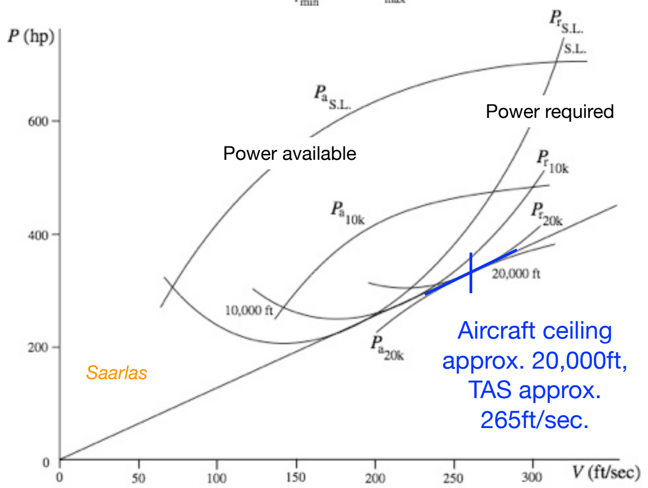
in order to obtain u.

<u>4. What is the maximum altitude the aircraft can attain?</u>

The power required (P_r) increases with altitude, while the power available (P_a) typically reduces with altitude. The aircraft's ceiling is at the match point.

If the available power is independent of airspeed and is a known function of σ, we can solve directly for ceiling. Otherwise use a numerical/graphical approach.





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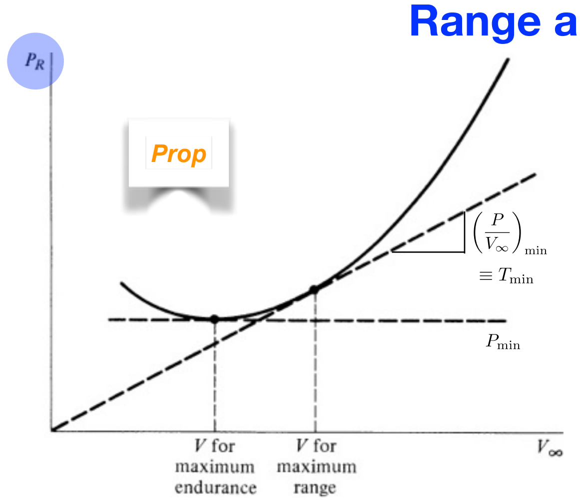
Range and endurance — 1



Propeller+piston, best characterised by power output

PSFC c = mass of fuel consumed per unit power per unit time.

<u>Maximum endurance: minimise kg of fuel per second</u>

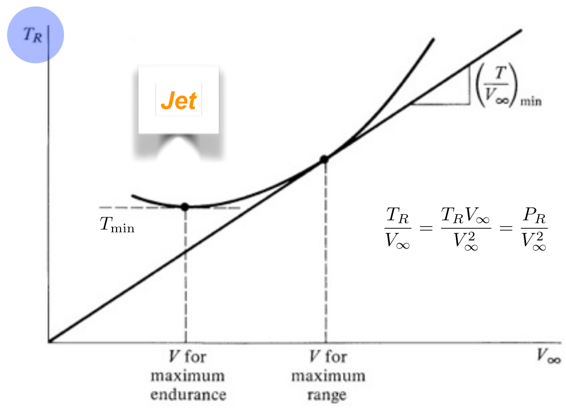
kg fuel/sec ∝ $c \times P_R$.

Fly at minimum-power speed, i.e. $(C_L^{3/2}/C_D)_{max}$.

<u>Maximum range: minimize kg of fuel per m travelled</u>

kg fuel/m ∝ (kg fuel/sec)/(m/sec) = $c \times P_R / V_\infty = c \times T_R$.

Fly at minimum-drag (and thrust) speed, V^*, $(C_L/C_D)_{max}$.



Jet, best characterised by thrust output

TSFC c_t = mass of fuel consumed per unit thrust per unit time.

<u>Maximum endurance: minimise kg of fuel per second</u>

kg fuel/sec ∝ $c_t \times T_R$.

Fly at minimum-drag (and thrust) speed, V^*, $(C_L/C_D)_{max}$.

<u>Maximum range: minimize kg of fuel per m travelled</u>

kg fuel/m ∝ (kg fuel/sec)/(m/sec) = $c_t \times T_R / V_\infty$.

Now $V_\infty = \sqrt{\frac{2}{\rho}\frac{W}{S}\frac{1}{C_L}}$ so $\frac{T_R}{V_\infty} = \frac{W}{V_\infty (C_L/C_D)} \propto \frac{1}{C_L^{1/2}/C_D}$

Fly at minimum power/kinetic energy speed, i.e. $(C_L^{1/2}/C_D)_{max}$.

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Range and endurance — 2

The ratios $C_L^{3/2}/C_D \propto L/DV$, $C_L/C_D \propto L/D$ and $C_L^{1/2}/C_D \propto VL/D$ can all be considered as functions of dimensionless speed $u=V/V^*$ where V^* is the minimum-drag speed.

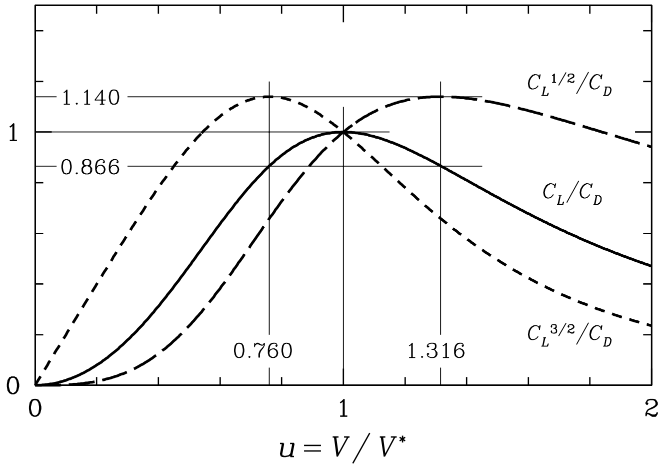
We already know

$$\frac{L}{D} = \left(\frac{C_L}{C_D}\right)^* \frac{2}{u^2+u^{-2}} = \left(\frac{C_L}{C_D}\right)^* \frac{2u^2}{u^4+1}$$

So $$\frac{L}{DV} \propto \frac{1}{u}\frac{2u^2}{u^4+1} = \frac{2u}{u^4+1}$$

and $$\frac{VL}{D} \propto u\,\frac{2u^2}{u^4+1} = \frac{2u^3}{u^4+1}$$

Plotting and then analysing these functions we can find various useful ratios associated with the maxima which are tabulated below.



Function	***Dimensionless***	***V/V*, at max***	**(L/D)/(L/D)*, at max**	**C_L/C_L^*, at max**
$L/(DV)$	$C_L^{3/2}/C_D$	$(1/3)^{1/4} = 0.760$	$(3/4)^{1/2} = 0.866$	$3^{1/2} = 1.732$
L/D	C_L/C_D	1	1	1
$(VL)/D$	$C_L^{1/2}/C_D$	$(3)^{1/4} = 1.316$	$(3/4)^{1/2} = 0.866$	$(1/3)^{1/2} = 0.577$

Recall: $C_L^* = (C_{D,0}/K)^{1/2}$; $(C_L/C_D)^* = 1/(4C_{D,0}K)^{1/2}$; $V^* = [(2/\rho)(W/S)(1/C_L^*)]^{1/2}$.

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Range and endurance — 3

Now we look at the specifics of estimating range and endurance.

Range; based on weight of fuel consumed per unit distance.

$$\frac{W_f}{x} = \mathrm{g}\frac{\mathrm{d}m}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \dot{m}\mathrm{g}\frac{\mathrm{d}t}{\mathrm{d}x} = -\frac{\dot{W}}{V} = -\frac{\mathrm{d}W}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = -\frac{\mathrm{d}W}{\mathrm{d}x} \equiv -\frac{\mathrm{d}W}{\mathrm{d}R}$$ where R is range. Hence $$\frac{\mathrm{d}R}{\mathrm{d}W} = -\frac{V}{\dot{m}\mathrm{g}}$$

Jet $$\dot{m}\mathrm{g} = \mathrm{g}\,c_t\,T \quad \text{and} \quad T = \frac{W}{L/D}$$

$$\frac{\mathrm{d}R}{\mathrm{d}W} = -\frac{V}{\mathrm{g}c_t\,T} = -\frac{1}{\mathrm{g}c_t}V\frac{L}{D}\frac{1}{W}$$ i.e. $$R = -\int_{W_i}^{W_f}\frac{1}{\mathrm{g}c_t}V\frac{L}{D}\frac{\mathrm{d}W}{W} = \int_{W_f}^{W_i}\frac{1}{\mathrm{g}c_t}V\frac{L}{D}\frac{\mathrm{d}W}{W}$$

where W_i is the initial, W_f is the final aircraft weight for a flight segment.

Now **assuming** c_t, V, L/D are all constants: $$R = \frac{1}{\mathrm{g}c_t}V\frac{L}{D}\ln\frac{W_i}{W_f}$$

$(3/4)^{1/2}$ $3^{1/4}$

We already know that VL/D is largest when we fly at $(C_L^{1/2}/C_D)_{max}$., i.e. L/D=0.866$(L/D)^*$ and V=1.316V^*.

$$R_{\max} = \left(\frac{27}{16}\right)^{1/4}\frac{1}{\mathrm{g}c_t}V^*\left(\frac{L}{D}\right)^*\ln\frac{W_i}{W_f} = \frac{1.140}{\mathrm{g}c_t}V^*\left(\frac{L}{D}\right)^*\ln\frac{W_i}{W_f}$$

Range and endurance — 4

Prop $\dot{m}g = gcP = \dfrac{gcDV}{\eta_{\rm pr}} = \dfrac{gcTV}{\eta_{\rm pr}} = \dfrac{gcWV}{\eta_{\rm pr}(L/D)}$ and $\dfrac{\mathrm{d}R}{\mathrm{d}W} = \left(-\dfrac{V}{\dot{m}g} = \right) - \dfrac{\eta_{\rm pr}(L/D)}{gcW}$

$\mathrm{d}R = -\dfrac{\eta_{\rm pr}}{gc}\dfrac{L}{D}\dfrac{\mathrm{d}W}{W}$ Hence, to maximise, assuming *L/D* and *c* const: $R_{\rm max} = \dfrac{\eta_{\rm pr}}{gc}\left(\dfrac{L}{D}\right)^* \ln\dfrac{W_i}{W_f}$

(as originally derived by Breguet).

<u>**Endurance**; based on weight of fuel consumed per unit time.</u>

$\dot{m}g = -\dot{W} = -\dfrac{\mathrm{d}W}{\mathrm{d}t} \equiv -\dfrac{\mathrm{d}W}{\mathrm{d}E}$ where *E* is endurance. Hence $\dfrac{\mathrm{d}E}{\mathrm{d}W} = -\dfrac{1}{\dot{m}g}$

Jet $\dot{m}g = g\,c_t\,T$ Then, working similarly to previously for jet range, we have

$\mathrm{d}E = -\dfrac{1}{gc_t}\dfrac{L}{D}\dfrac{\mathrm{d}W}{W}$ Assuming c_t and *L/D* const: $E = \dfrac{1}{gc_t}\dfrac{L}{D}\ln\dfrac{W_i}{W_f}$ $E_{\rm max} = \dfrac{1}{gc_t}\left(\dfrac{L}{D}\right)^* \ln\dfrac{W_i}{W_f}$

Prop As above: $\dot{m}g = gcP = \dfrac{gcDV}{\eta_{\rm pr}} = \dfrac{gcTV}{\eta_{\rm pr}} = \dfrac{gcWV}{\eta_{\rm pr}(L/D)}$

$\mathrm{d}E = -\dfrac{\eta_{\rm pr}}{gc}\dfrac{1}{V}\dfrac{L}{D}\dfrac{\mathrm{d}W}{W}$ $E = \dfrac{\eta_{\rm pr}}{gc}\dfrac{1}{V}\dfrac{L}{D}\ln\dfrac{W_i}{W_f}$ which we know is maximised when flying at $(C_L^{3/2}/C_D)_{\rm max}$,

where *L/D* = 0.866(*L/D*)*, *V* = 0.760*V**: $E_{\rm max} = \left(\dfrac{27}{16}\right)^{1/4}\dfrac{\eta_{\rm pr}}{gc}\dfrac{1}{V^*}\left(\dfrac{L}{D}\right)^* \ln\dfrac{W_i}{W_f}$

$(3/4)^{1/2}$ $(1/3)^{1/4}$

NB: In a design problem, we will normally have the range or endurance specified and need to find the weight fraction W_i/W_f and from this $W_{\rm fuel} = W_i\,(1-W_f/W_i)$.

Range and endurance — 5

$V^* = \sqrt{\dfrac{2}{\rho}\dfrac{W}{S}\dfrac{1}{C_L^*}}$

Summary

Optimal conditions for range and endurance all demand flying at a fixed point on the drag polar.
This leads to varying implications for optimal flight strategy as fuel is consumed and *W* falls.

<u>**Range, *R***</u>

Type	**Equation**	**Optimal flight strategy**
Jet	$R = \dfrac{1}{gc_t}V\dfrac{L}{D}\ln\dfrac{W_i}{W_f}$	Maximized at high altitude (high *V* ⇒ low ρ). Fly at 1.316*V**, 0.577C_L*, 0.866(C_L/C_D)*. <u>If M_{DD} > 1.316*V**,</u> fly at M_{DD} and close to C_L*, (C_L/C_D)*.
Prop	$R = \dfrac{\eta_{\rm pr}}{gc}\dfrac{L}{D}\ln\dfrac{W_i}{W_f}$	Independent of altitude. Fly at *V**, C_L*, (C_L/C_D)*.

<u>**Endurance, *E***</u>

Type	**Equation**	**Optimal flight strategy**
Jet	$E = \dfrac{1}{gc_t}\dfrac{L}{D}\ln\dfrac{W_i}{W_f}$	Independent of altitude. Fly at *V**, C_L*, (C_L/C_D)*.
Prop	$E = \dfrac{\eta_{\rm pr}}{gc}\dfrac{1}{V}\dfrac{L}{D}\ln\dfrac{W_i}{W_f}$	Maximized at low altitude (low *V* ⇒ high ρ). Fly at 0.760*V**, 1.732C_L*, 0.866(C_L/C_D)*. *V** reduces as *W* falls, *h* fixed near SL.

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Unaccelerated climbing flight

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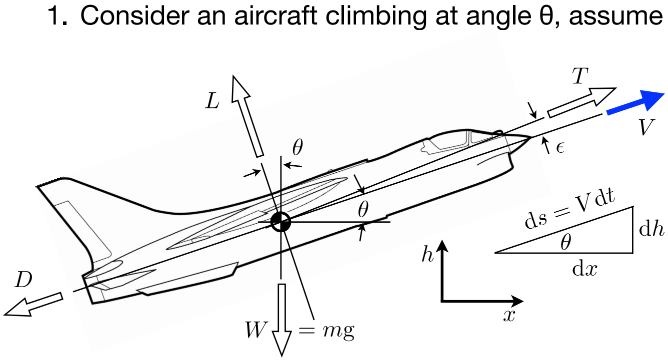
Climb rate and angle — 1

1. Consider an aircraft climbing at angle θ, assume thrust angle $\epsilon = 0$.



The fundamental performance equation, obtained by dynamical equilibrium tangential to the flight path:

$$\frac{(T-D)V}{W} = V\sin\theta + V\frac{\dot{V}}{g} = \frac{d}{dt}\left(h + \frac{V^2}{2g}\right) = \frac{de}{dt}$$

Rearrange, using $d(V^2)/dt = 2V \times dV/dt$ and $dV/dt = dV/dh \times dh/dt$:

An equation for (weight-) specific excess power.

$$\frac{dh}{dt} = \frac{(T-D)V}{W}\left[\frac{1}{1+\frac{V}{g}\frac{dV}{dh}}\right]$$

where $dh/dt = V_V = V\sin\theta$ is the rate of climb.

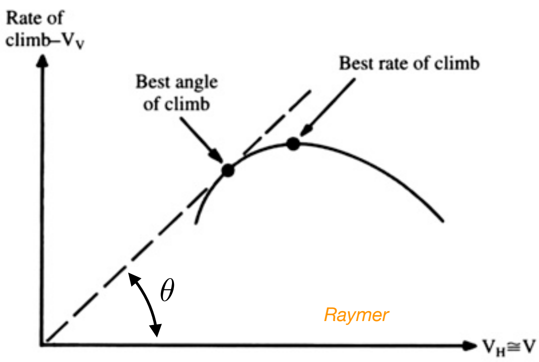
The components of forces normal to the flight path gives, when θ=const, $L = W\cos\theta$ $\quad n = \frac{L}{W} = \cos\theta$

In the simplified treatment to be given here, $V \times dV/dh$ is assumed small and $\cos\theta \to 1$.
Using just the first of these we have

Rate of climb $\quad \frac{dh}{dt} = V_V = \frac{(T-D)V}{W} = \frac{P-DV}{W}$

Angle of climb $\quad \frac{V_V}{V} = \sin\theta = \frac{T-D}{W} = \frac{P-DV}{WV}$

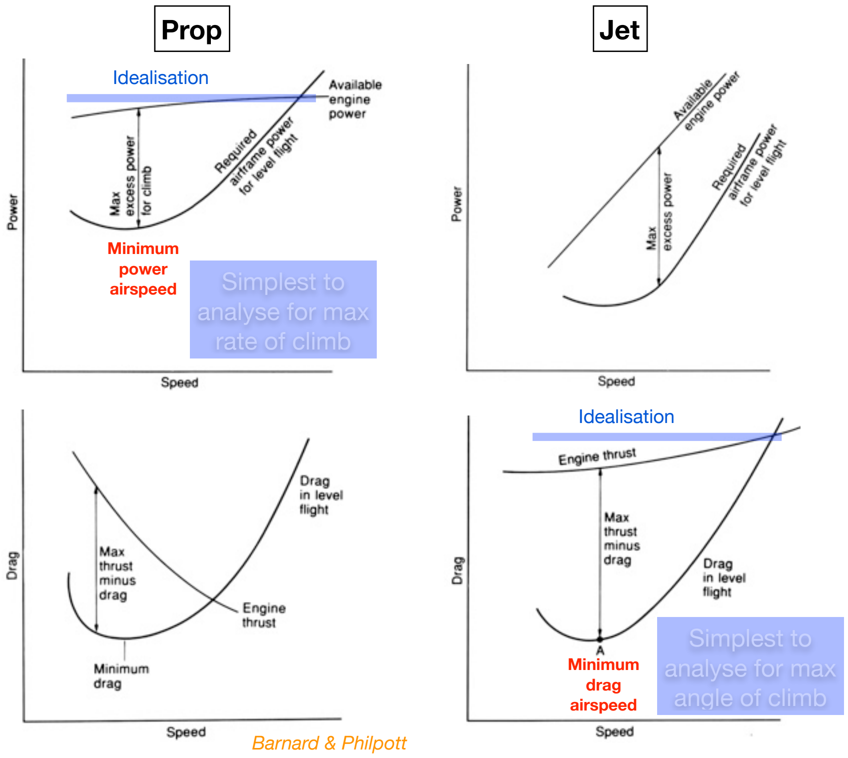
These are two related variables whose maxima occur at different airspeeds.



Climb rate and angle — 2

Rate of climb $\frac{\mathrm{d}h}{\mathrm{d}t} = V_V = \frac{(T-D)V}{W} = \frac{P-DV}{W}$ simplest to analyse if *P* independent of *V*

Angle of climb $\frac{V_V}{V} = \sin\theta = \frac{T-D}{W} = \frac{P-DV}{WV}$ simplest to analyse if *T* independent of *V*



Barnard & Philpott

When we use idealised approximations for propulsion and drag we find that <u>propeller-driven aircraft are simplest to analyse for rate-of climb performance</u> while <u>jet-propelled aircraft are simplest to analyse for angle of climb</u>.

If we also assume that cosθ→1 then we find that <u>maximum rate of climb for a propeller-driven aircraft occurs at the minimum-power airspeed</u>, while the <u>maximum angle of climb for a jet-propelled aircraft occurs at the minimum-drag airspeed</u>.

The simplification cosθ→1 generally leads to only small errors unless *T/W*~1.

Climb rate and angle — 3

To make progress we unify and non-dimensionalize the treatment starting from the FPE.

We had already assumed that *V*×d*V*/d*h* = 0. For now we will retain cosθ. Working with angle of climb:

$\frac{T-D}{W} = \sin\theta$, thrust producing engines, or, for power-producing: $\frac{P_A - DV}{WV} = \sin\theta$ where $P_A = \eta_{\mathrm{pr}} P_{\mathrm{es}}$

Say we can have either or both of thrust, power-producing engines:

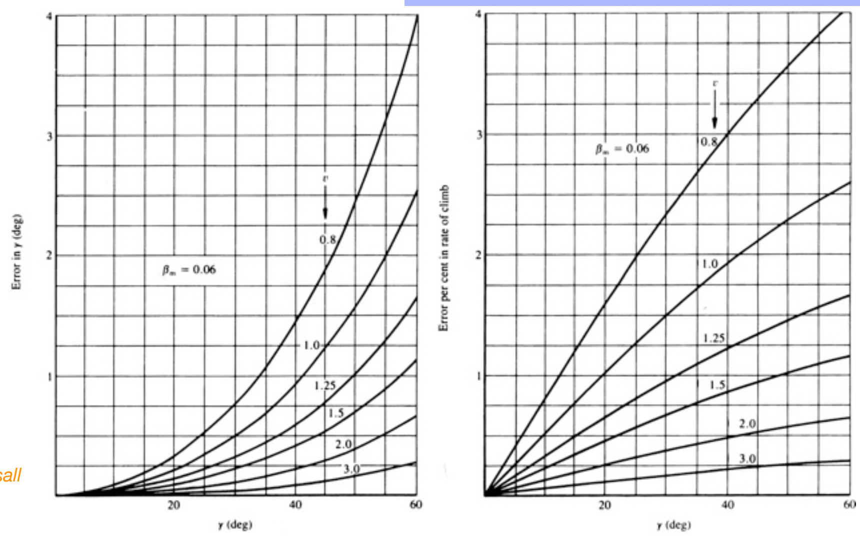
$$\frac{\eta_{\mathrm{pr}} P_{\mathrm{es}}}{V} + T - D = W\sin\theta = \frac{L}{n}\sin\theta$$ (using *L* = *nW*). ***FPE,*** with *V*×d*V*/d*h*=0.

Now use *V*=*uV** and *D/D** = ½[u²+*n*²/*u*²], with *D** = ½ρ*V**²*S*×2*C*$_{D,0}$:

$$\frac{\eta_{\mathrm{pr}} P_{\mathrm{es}}}{uV^* D^*} + \frac{T}{D^*} - \frac{1}{2}\left[u^2 + \frac{n^2}{u^2}\right] = \frac{1}{n}\frac{L}{D^*}\sin\theta$$ or equivalently $$\frac{\lambda}{u} + \tau - \frac{1}{2}\left[u^2 + \frac{n^2}{u^2}\right] = \frac{1}{n}\left(\frac{L}{D}\right)^* \sin\theta$$

in which we can set either λ or τ to zero as we wish (for jet or prop).

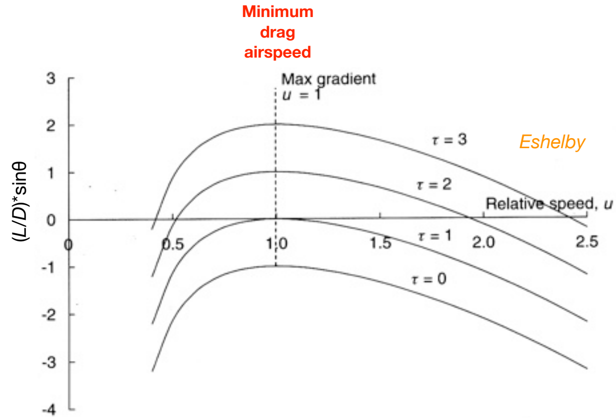
From this point we will take *n* = 1, i.e. assume cosθ→1. Typically the resulting errors are fairly small until θ > 45°, see plots:



Mair & Birdsall

Climb rate and angle — 4

Jet i.e. thrust-producing engines (λ=0)

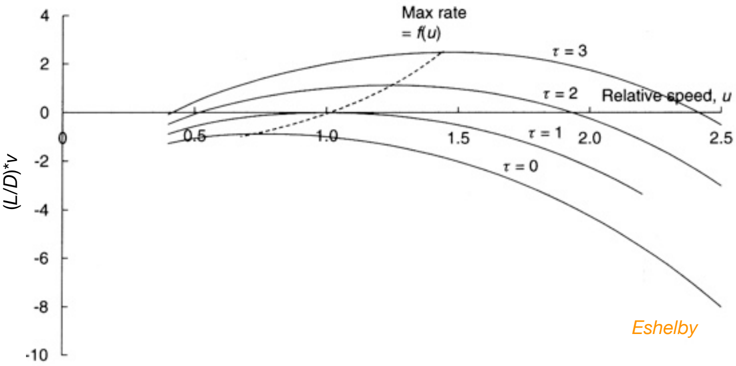


1. Climb gradient $\tau - \frac{1}{2}\left[u^2 + u^{-2}\right] = \left(\frac{L}{D}\right)^* \sin\theta$

Sinθ, hence θ, is maximized where ½[u^2+u^{-2}] is minimized, which occurs when u=1:

$\tau - 1 = \left(\frac{L}{D}\right)^* \sin\theta$ or $\theta_{\max} = \sin^{-1}\left[\frac{\tau - 1}{(L/D)^*}\right]$

2. Rate of climb $\tau u - \frac{1}{2}\left[u^3 + u^{-1}\right] = \left(\frac{L}{D}\right)^* v$ where $v = V_V/V^*$ and $\sin\theta = V_V/V$,



which has a turning point at

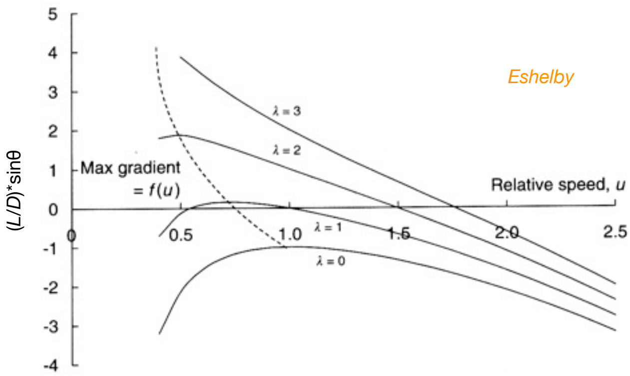
$$u = \frac{1}{\sqrt{3}}\left[\tau + \left(\tau^2 + 3\right)^{1/2}\right]^{1/2}$$

Making the substitution $z = 1 + \left(1 + 3/\tau^2\right)^{1/2}$ we obtain

$$v_{\max} = \frac{T}{W}\left(\frac{\tau z}{3}\right)^{1/2}\left[1 - \frac{z}{6} - \frac{3}{2\tau z}\right]$$

Climb rate and angle — 5

Prop i.e. power-producing engines (τ=0)



1. Climb gradient: the gradient is given by

$$\left(\frac{L}{D}\right)^* \sin\theta = \frac{\lambda}{u} - \frac{1}{2}\left[u^2 + u^{-2}\right]$$

which has a turning point where

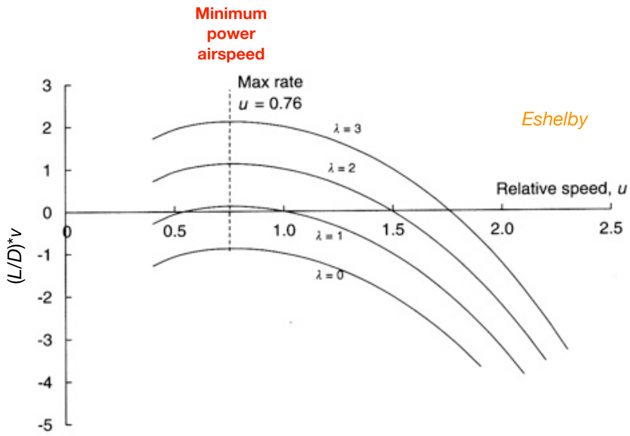
$$u^4 + \lambda u - 1 = 0$$

requiring numerical solution.

The gradient is zero at $\lambda = \left(\frac{16}{27}\right)^{1/4}$ & $u = \left(\frac{1}{3}\right)^{1/4}$

I.e. minimum power for level flight.

The maximum occurs for lower u as λ increases and eventually the aircraft will stall before maximum climb gradient is reached.



2. Rate of climb: maximized where

$$\left(\frac{L}{D}\right)^* v = \lambda - \frac{1}{2}\left[u^3 + u^{-1}\right]$$

has a turning point at $u = \left(\frac{1}{3}\right)^{1/4} = 0.760$

leading to

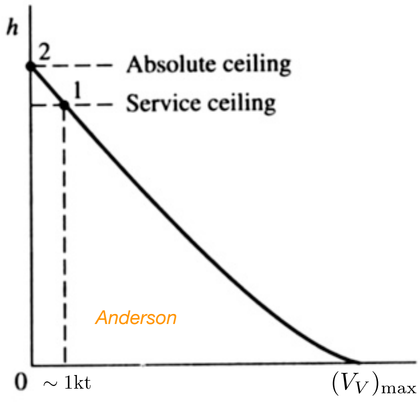
$$v_{\max} = \frac{1}{(L/D)^*}\left[\lambda - \left(\frac{16}{27}\right)^{1/4}\right]$$

$$= \frac{1}{(L/D)^*}\left(\frac{27}{16}\right)^{1/4}\left[\frac{P_A}{P_{\min}} - 1\right]$$

Climb rate and angle — 6

1. Absolute ceiling — altitude where maximum rate of climb falls to zero.
2. Service ceiling — altitude where maximum rate of climb falls to (e.g)

   100 ft/min (1kt, 0.508 m/s): JAR25 for Propeller aircraft
   500 ft/min (5kt, 2.540 m/s): JAR25 for Jet aircraft



We already know that the absolute ceiling is where the specific excess thrust (or power, for prop) is zero in level flight.

For service ceiling we have

Jet (thrust independent of speed)

$$(V_V)_{\max} = \left[\frac{(W/S)z}{3\rho C_{D,0}}\right]^{1/2}\left(\frac{T}{W}\right)^{3/2}\left[1-\frac{z}{6}-\frac{3}{2(T/W)^2(L/D)^2_{\max}z}\right]$$

Prop+engine (power independent of speed)

$$(V_V)_{\max} = \frac{\eta_{\mathrm{pr}}P}{W} - \frac{2}{\rho}\sqrt{\frac{K}{3C_{D,0}}}\left(\frac{W}{S}\right)^{1/2}\frac{1.155}{(L/D)_{\max}}$$

As previously defined $z = 1+(1+3/\tau^2)^{1/2} = 1+\left(1+\dfrac{3}{(L/D)^2_{\max}(T/W)^2}\right)^{1/2}$

Procedure for service ceiling estimate: take one or other of these equations and solve for the density ρ at which $(V_V)_{\max}$=100 ft/min, either numerically (if we have climb thrust or power as a function of σ, say) or graphically (in the more general case) Then work back to get *h*, from ISA.

It is best to compute and plot $(V_V)_{\max}$ as a function of *h.*

Climb rate and angle — 7

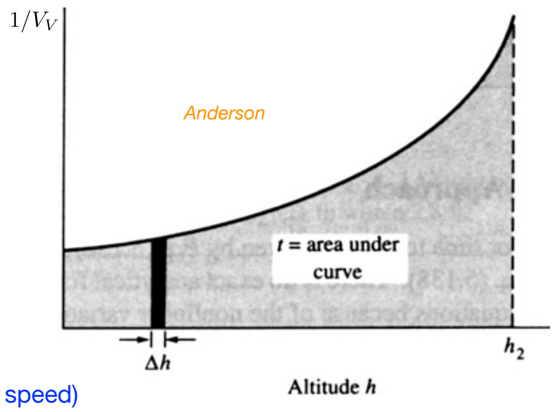
3. Time to climb to a given height can be obtained by integration.

$$\frac{\mathrm{d}h}{\mathrm{d}t} = V_V, \quad \text{i.e.} \quad \mathrm{d}t = \frac{\mathrm{d}h}{V_V}$$

$$t = \int_{h_1}^{h_2}\frac{\mathrm{d}h}{V_V}$$

$$t_{\min} = \int_{h_1}^{h_2}\frac{\mathrm{d}h}{(V_V)_{\max}}$$

$$t_{\min} = \int_{0}^{h_2}\frac{\mathrm{d}h}{(V_V)_{\max}}$$



Jet (note that we can include thrust variations with height and speed)

$$(V_V)_{\max} = \left[\frac{(W/S)z}{3\rho C_{D,0}}\right]^{1/2}\left(\frac{T}{W}\right)^{3/2}\left[1-\frac{z}{6}-\frac{3}{2(T/W)^2(L/D)^2_{\max}z}\right]$$

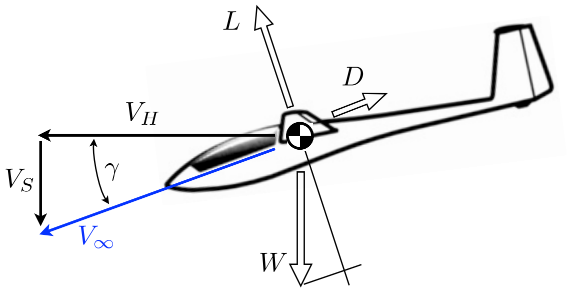
Prop+engine (can include power variations with height and speed)

$$(V_V)_{\max} = \frac{\eta_{\mathrm{pr}}P}{W} - \frac{2}{\rho}\sqrt{\frac{K}{3C_{D,0}}}\left(\frac{W}{S}\right)^{1/2}\frac{1.155}{(L/D)_{\max}}$$

Procedure: take one or other of these equations, and with density ρ at each altitude *h*, compute $1/(V_V)_{\max}$. Integrate from given starting height to desired height to get $t_{\min}$.

Maximum range and endurance in gliding flight

Gliding flight is a special case where there is no engine power supplied. Gravitational potential energy is dissipated to drag power. V_V is negative and is replaced with sink speed V_S.



Force equilibrium

$$L = W\cos\gamma \qquad D = W\sin\gamma \quad \longrightarrow \quad V_\infty = \sqrt{\frac{\rho}{2}\frac{W}{S}\cos\gamma\frac{1}{C_L}}$$

Geometry

$$\tan\gamma = \frac{V_S}{V_H} = \frac{\Delta z}{\Delta t}\frac{\Delta t}{\Delta x} = \frac{\Delta z}{\Delta x} = \frac{D}{L}$$

1. To maximize range for a given height loss (minimize γ), maximize L/D. **i.e. fly @ $(C_L/C_D)_{max}$**

2. To maximize endurance for a given height loss (or, maximize rate of climb in rising air), minimize V_S.

$$V_S = V_\infty\sin\gamma = V_\infty\frac{D}{W} = V_\infty\tfrac{1}{2}\rho V_\infty^2\frac{S}{W}C_D = \frac{\rho}{2}\frac{S}{W}\left(\frac{\rho}{2}\frac{W}{S}\cos\gamma\frac{1}{C_L}\right)^{3/2}C_D = \left(\frac{\rho}{2}\frac{W}{S}\right)^{1/2}(\cos\gamma)^{3/2}\frac{C_D}{C_L^{3/2}}$$

Typically, glide angle γ is small, so assume $\cos\gamma \to 1 \;\Rightarrow$

$$V_S \approx \left(\frac{\rho}{2}\frac{W}{S}\right)^{1/2}\frac{C_D}{C_L^{3/2}}$$

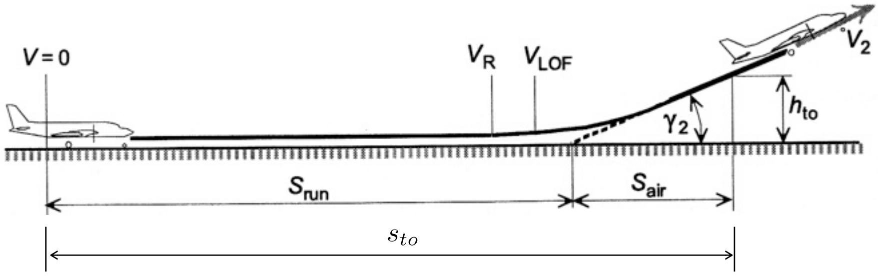
For larger glide angles, use $\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$

i.e. fly @ $(C_L^{3/2}/C_D)_{max}$

Takeoff and landing

Normal take-off

A normal takeoff is made with all engines operating.



Torenbeek & Wittenberg

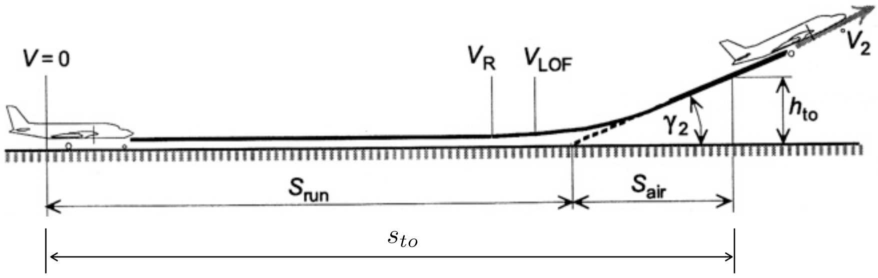
1. During the ground run the aircraft rolls from standstill with nosewheel on the ground so that the angle of attack is nearly constant (and typically a value that makes aerodynamic drag small).
2. Rotation is initiated at speed V_R. When $L = W$, the aircraft becomes airborne at speed V_{LOF}.
3. During the initial airborne phase $L > W$ and there is acceleration normal to the flight path, but soon afterwards the angle of climb settles to a constant value and the undercarriage is retracted to reduce drag. The aircraft accelerates to "safety speed" V_2 which is at least k_{to} (usually a factor of 1.2) times the stall speed.
4. Takeoff is said to be completed after the aircraft reaches a "screen height" h_{to}, which is 35 ft for commercial aircraft and 50 ft for military aircraft.
5. The total runway length must exceed the ground roll plus the distance required to clear the screen.

$$s_{\text{to}} = s_{\text{run}} + s_{\text{air}}$$

$$V_2 > k_{\text{to}} V_{\text{stall}} = 1.2\sqrt{\frac{2}{\rho}\frac{W}{S}\frac{1}{C_{L,\text{max}}}} \qquad C_{L_2} \le C_{L,\text{max}}/1.44$$

Normal take-off

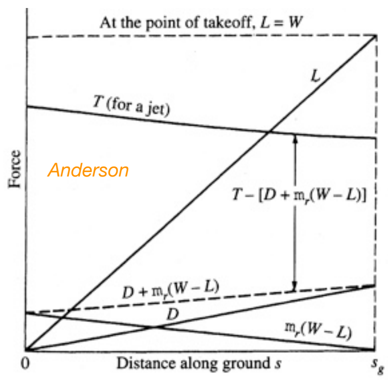
Simplified analysis assumes $V_{LOF} = V_2$ and that a steady climb with speed V_2 and angle γ_2 starts at lift-off.



Torenbeek & Wittenberg

$$s_{\text{to}} \approx \frac{V_2^2}{2\bar{a}} + \frac{h_{\text{to}}}{\tan\gamma_2}$$

where $\bar{a}$ is the average acceleration over the ground.



Anderson

A more complete analysis has to account for a drop in engine thrust with speed, the friction of the tyres on the ground, and the fact that the vertical reaction force (hence friction force) falls as the aircraft starts to produce lift even while on the ground.

For now we take these complexities into account by applying a simple reduction factor on thrust and say that

$$\bar{a} = r_T \frac{T_{\text{to}}}{m} \qquad \text{or} \qquad \bar{a} = r_T \frac{T_{\text{to}}}{W} g$$

where for a jet aircraft $r_T \approx 0.8 - 0.9$ but is lower again for propeller aircraft where thrust falls with speed.

Normal take-off

$$s_{to} \approx \frac{V_2^2}{2\bar{a}} + \frac{h_{to}}{\tan\gamma_2} \qquad \bar{a} = r_T \frac{T_{to}}{W} g$$

s_{run} s_{air}

Recall for steady climb $\sin\gamma_2 = \frac{T-D}{W} = \frac{T}{W} - \frac{D}{W}$ where $W \to L = \frac{1}{2}\rho V_2^2 S C_{L_2}$ or $V_2^2 = \frac{2}{\rho}\frac{W}{S}\frac{1}{C_{L_2}}$

$$\sin\gamma_2 = \left(\frac{T}{W} - \frac{C_D}{C_L}\right)_2 \qquad C_{L_2} \approx C_{L,\max}/k_{to}^2 \quad \text{and} \quad C_D = C_{D,0} + KC_{L_2}^2$$

and $\sin\gamma_2 \to \tan\gamma_2$

finally

$$s_{to} \approx \frac{1}{\rho g r_T}\frac{1}{C_{L_2}}\frac{W}{S}\frac{W}{T_{to}} + \frac{h_{to}}{T_2/W - (C_D/C_L)_2}$$

s_{run} s_{air}

Typically an additional 15% safety margin is added to this value (or any better estimate).

Note that T_2 may be significantly smaller than T_{to}, especially so for a propeller aircraft.

1. Ground run s_{run} increases quadratically with weight W and is reduced by either decreasing the wing loading W/S or increasing the thrust/weight ratio T_{to}/W, or both. Increasing weight also increases the air distance s_{air}.
2. Air density+temperature may have a significant effect on both ρ and T. High, hot take-offs are worst.
3. Increasing flap deflections will increase $C_{L\max}$ and hence C_{L2}, which reduces the first term but increases C_D/C_L and hence the second term. There is an optimum flap deflection which is typically less than the value used at landing, and hence $C_{L,\max,to} < C_{L,\max,land}$.

Engine failure during take-off

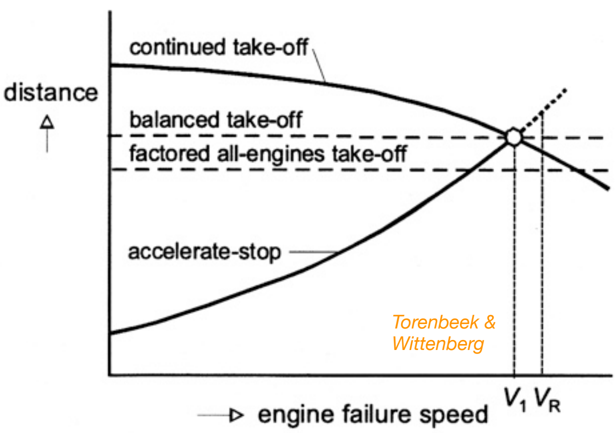
Engine failure (fortunately rare) will obviously lead to an aborted takeoff in a single-engine aircraft.

Multi-engined aircraft are designed so that they will still be able to climb with ***one engine inoperative*** (**OEI**).

If an engine fails then obviously total thrust is reduced but also the unpowered engine may contribute significant drag. There is typically a lateral asymmetry of thrust (and associated yawing moment) and the available control surface authority (as well as the pilot!) must be able to cope with this.

If engine failure is recognised at a sufficiently low speed (on the ground) then all engines are throttled back and all available braking (excluding thrust reversal) is used to de-accelerate the aircraft to a halt. The associated total distance on the runway is called the ***accelerate-stop distance***.

After a certain speed is reached it takes less runway distance to continue the takeoff and climb to screen height h_{to} than it does to brake to a halt. This speed, less than the rotation speed V_R, is called the ***decision speed*** V_1. Above V_1 the pilot will continue to take-off regardless of the runway length available.



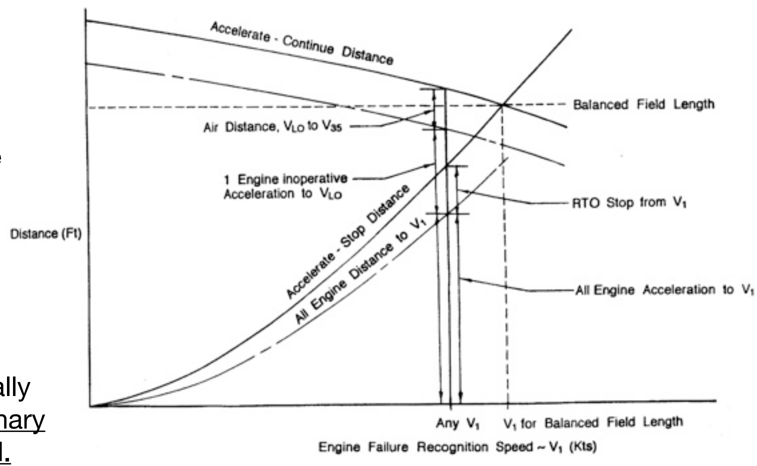
For any given engine failure speed, the total distance required required to accelerate to it and stop can be found, as can the distance required to continue and climb over screen height. The first increases with speed while the second decreases. The distance at which they are equal is called the ***balanced field length*** (**BFL**) and the associated speed is V_1.

The required runway length is the minimum of the BFL and 1.15 times the value estimated for the all-engines-operative case.

Because the aircraft's climbing capacity is reduced, there is typically also a regulated minimum climb gradient at V_2 with OEI. This is 2.4% for two engines, 2.7% for 3 engines and 3.0% for four engines.

TOP & TOFL

Schaufele



For transport aircraft, takeoff field length (TOFL) is specified to FAR/JAR rules for "balanced take-off field length" which is the length of takeoff field at which the space required to accelerate, recognise an engine failure, and brake to a halt, is the same as the space required to continue to take-off with one engine inoperative. This is the required TOFL unless 1.15× the distance to takeoff with all engines operating is greater, in which case that is the TOFL.

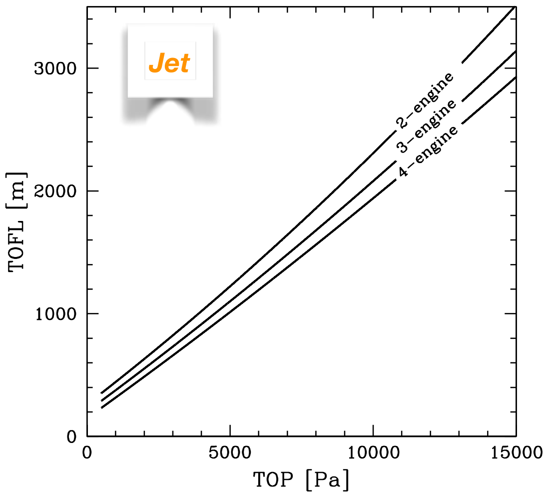
Detailed computation of these lengths is usually handled by a computer program. For preliminary design, correlations are typically used instead.

The correlating parameter is called the TOP (takeoff parameter). It combines wing loading, thrust or power loading, air density and $C_{L\max}$ in take-off configuration in a sensible way: the larger TOP, the larger TOFL must be. TOP is a dimensional parameter, and note its dimensions and units are different for jet and propeller aircraft.

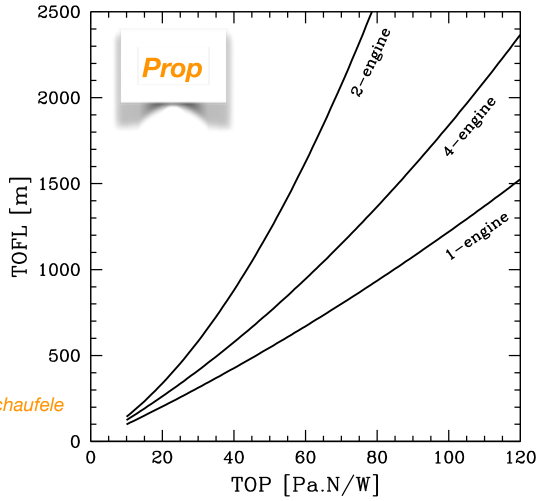
$$\mathrm{TOP_{jet}} = \frac{W_0}{T_{0.7}}\frac{W_0}{S}\frac{1}{\sigma}\frac{1}{C_{L\,\max,TO}} \qquad \mathrm{TOP_{prop}} = \frac{W_0}{P_0}\frac{W_0}{S}\frac{1}{\sigma}\frac{1}{C_{L\,\max,TO}}$$

For jet aircraft, thrust is evaluated (using low-M model for α) at 0.70 V_{TO}, where V_{TO} is typically 1.2 V_{stall}. For propeller aircraft, the maximum sea-level power is used.

TOP & TOFL



Adapted from Schaufele



12. The following curve fits link TOP to TOFL. Note again that the correlations relate dimensional variables.

Jet

Two-engine jet aircraft: $\mathrm{TOFL,\ m} = 261.3 + 0.1800 \times \mathrm{TOP} + 2.460 \times 10^{-6} \times \mathrm{TOP}^2$

Three-engine jet aircraft: $\mathrm{TOFL,\ m} = 203.6 + 0.1713 \times \mathrm{TOP} + 1.635 \times 10^{-6} \times \mathrm{TOP}^2$

Four-engine jet aircraft: $\mathrm{TOFL,\ m} = 148.6 + 0.1668 \times \mathrm{TOP} + 1.236 \times 10^{-6} \times \mathrm{TOP}^2$

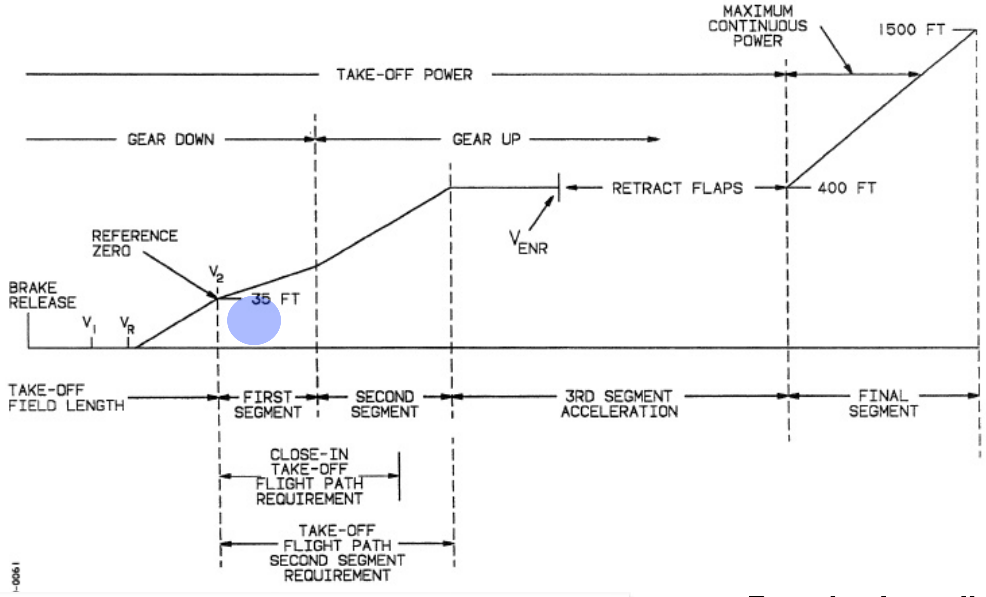
Prop

One-engine prop aircraft: $\mathrm{TOFL,\ m} = 9.65 \times \mathrm{TOP} + 0.0255 \times \mathrm{TOP}^2$

Two-engine prop aircraft: $\mathrm{TOFL,\ m} = 11.8 \times \mathrm{TOP} + 0.255 \times \mathrm{TOP}^2$

Four-engine prop aircraft: $\mathrm{TOFL,\ m} = 11.8 \times \mathrm{TOP} + 0.066 \times \mathrm{TOP}^2$

Four Climb Segments after TO, JAR25 regulations



Second segment is usually the most demanding part of climb so far as aircraft performance is concerned.

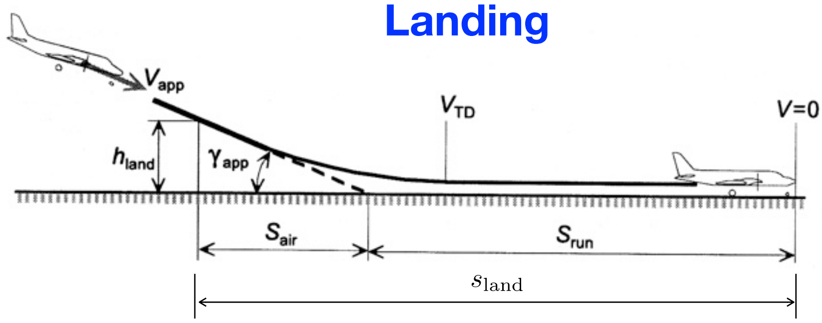
$$V_2 = k_{\text{TO}} V_{\text{stall, TO Config}} = 1.2 V_{\text{stall, TO Config}}$$

$$\gamma_2 \approx \left(\frac{T_{\text{OEI}}}{W_0} - \frac{C_D}{C_L} \right)_2$$

Required gradient, %, OEI

No. of Engines	4	3	2
1st Segment	0.5	0.3	0.0
2nd Segment	3.0	2.7	2.4
3rd Segment	0	0	0
Final Segment	1.7	1.4	1.1

Landing



Torenbeek & Wittenberg

1. During *landing approach* the aircraft flies at a steady speed $V_{app} > k_{app}\, V_{stall} = 1.3\, V_{stall}$. The gradient γ_{app} is typically around 2.5° to 3° for commercial aircraft.
2. Once the *runway threshold height* h_{land} (typically 50 ft or 15 m) is reached the engines are throttled back and the pilot executes a *landing flare* or *round-out* to touch-down at V_{td}, typically 1.15 V_{stall}.
3. After touch-down of all the undercarriage elements the aircraft is slowed by wheel brakes (and perhaps airbrakes) until it comes to rest. While engine thrust reversal may be applied this is typically not included in an analysis designed to compute the minimum runway length required.

For a simplified analysis, $s_{\text{land}} = s_{\text{air}} + s_{\text{run}} \approx \dfrac{h_{\text{land}}}{\tan \gamma_{\text{app}}} + \dfrac{V_{\text{app}}^2}{2|\bar{a}|}$ where $V_{\text{app}} = 1.3 \sqrt{\dfrac{2}{\rho} \dfrac{W}{S} \dfrac{1}{C_{L,\max}}}$

leading to

$$s_{\text{land}} \approx \frac{h_{\text{land}}}{\tan \gamma_{\text{app}}} + 1.69 \frac{W/S}{\rho |\bar{a}| C_{L,\max}}$$

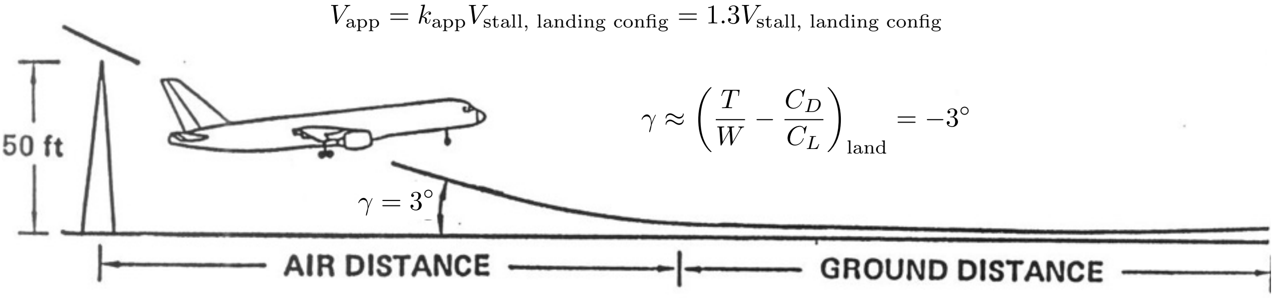
Note that the wing loading *W/S* may be much less than the maximum takeoff value, owing to fuel use.

Max de-acceleration possible on a dry concrete runway is $|\bar{a}|/\text{g} \equiv \mu = 0.3 \text{ to } 0.5$, use 0.15 for wet.

Note also that, unlike the case for take-off, the thrust loading *T/W* does not come into account here.

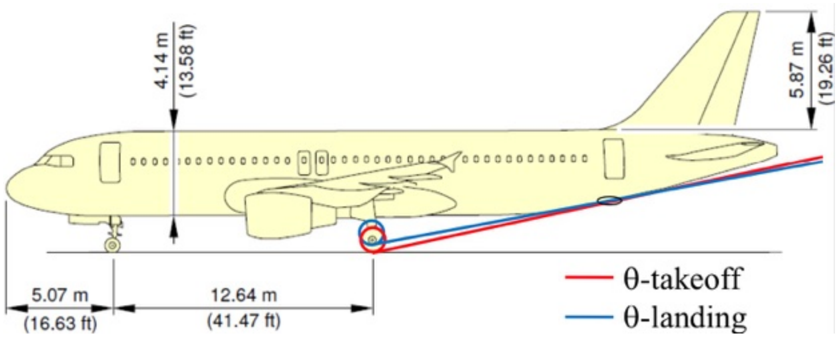
Landing approach

$$V_{\text{app}} = k_{\text{app}} V_{\text{stall, landing config}} = 1.3 V_{\text{stall, landing config}}$$



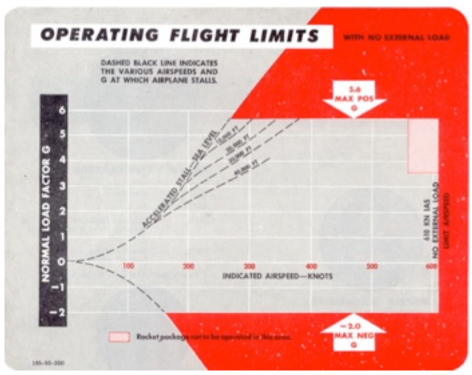
$$\gamma \approx \left(\frac{T}{W} - \frac{C_D}{C_L} \right)_{\text{land}} = -3°$$

Note that fuselage upsweep/clearance and landing gear length must be adequate to avoid ground impact accounting for a 3° glide slope when trimmed for AOA at approach C_L.



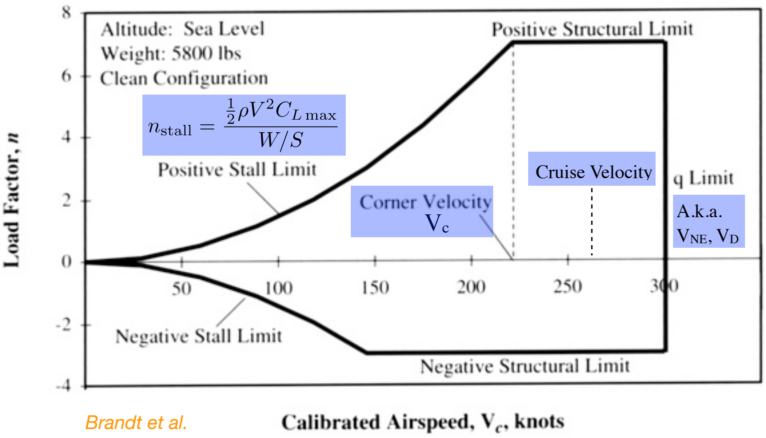
It is usual to leave 2.5° additional clearance, or 1° with a tail skid.

Manoeuvring flight



The V—n diagram

1. This expresses the load-factor/speed envelope of the aircraft as determined by performance constraints (e.g. stall) and structural strength. Its limits vary with altitude and aircraft loading.
2. The load factor is derived from $L=nW$, i.e. $n=L/W$ and describes how much load the structure carries compared to the case in level flight.



Brandt et al.

3. Normal level flight has n=1.
4. Exceeding the structural limit n value can lead to airframe damage or breakage.
5. Exceeding the dynamic pressure (q) limit can lead to flutter or shock buffet.
6. Typically, positive structural limits are larger than the negative limits.
7. At the 'corner velocity' V_c, simultaneously at the structural strength and aerodynamic stall limits, the maximum rate of turn is achieved.

$$V_c = \sqrt{\frac{2}{\rho}\frac{W}{S}\frac{n_{\text{limit}}}{C_{L\,\max}}}$$

8. Example codified limit load factors:

McCormick

TABLE 10.1: Maximum load factors for various aircraft based on FAR-25 and 23.

Aircraft Type	Load Factor
General Aviation (normal)	$-1.25 \le n \le 3.1$
General Aviation (utility)	$-1.8 \le n \le 4.4$
General Aviation (acrobatic)	$-3.0 \le n \le 6.0$
Homebuilt	$-2 \le n \le 5$
Commercial Transport	$-1.5 \le n \le 3.5$
Fighter	$-4.5 \le n \le 7.75$

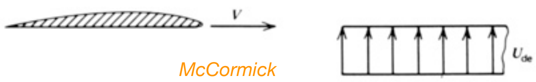
*** Regulations typically require an additional structural safety factor of approx. 1.5 at the peak load factors.**

TABLE 10.2: Load factors for transport aircraft based on FAR-25.

W_{TO}(lbs)	n_{max}
$\le$4100	3.8
$4100 < W_{TO} \le 50{,}000$	$2.1 + (24{,}000/(W_{TO} + 10{,}000))$
>50,000	2.5

The gust load diagram

1. Allowance is made for atmospheric turbulence in the form of gust loading factors, using gust velocities based on statistics and experience, varying with altitude.
2. Consider an aircraft encountering an idealised gust, speed U_{de}, in level flight:



McCormick

$$\Delta\alpha \approx \frac{U_{\text{de}}}{V}, \quad \Delta L = \tfrac{1}{2}\rho V_\infty^2 S \frac{\partial C_L}{\partial \alpha}\frac{U_{\text{de}}}{V_\infty} \qquad W = \tfrac{1}{2}\rho V_\infty^2 S \frac{\partial C_L}{\partial \alpha}\alpha$$

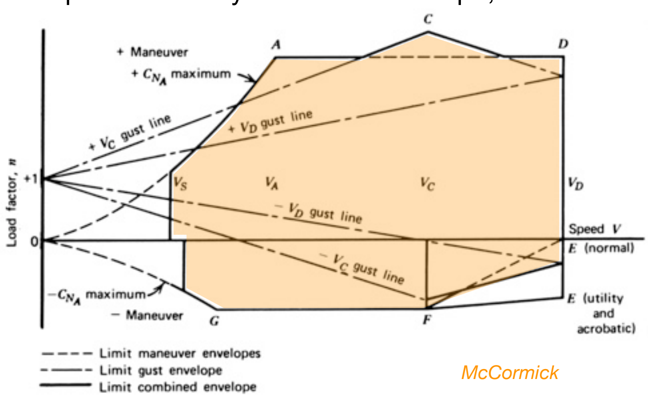
$$n = \frac{L}{W} = \frac{W + \Delta L}{W} = 1 + \frac{U_{\text{de}}}{V_\infty \alpha} = 1 + \frac{\rho V_\infty U_{\text{de}}(\partial C_L/\partial\alpha)}{2(W/S)}$$

3. A `gust alleviation factor' K_g is applied to allow for aircraft motion/flexure in gust:

$$n = 1 + \frac{K_g \rho V_\infty U_{\text{de}}(\partial C_L/\partial\alpha)}{2(W/S)}$$

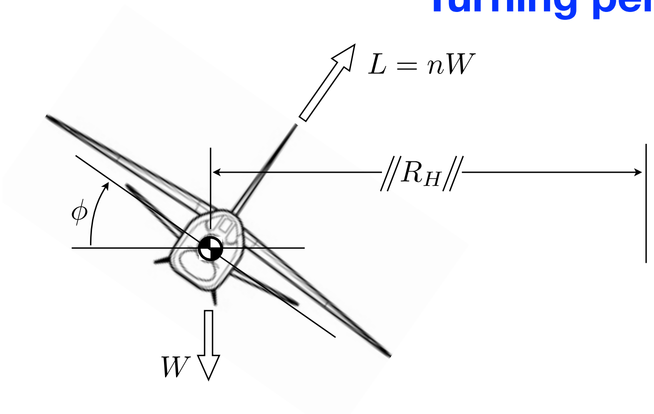
K_g is a quasi-empirical function of aircraft density relative to air density.

4. Different gust factors are applied at different flight speeds V_C, V_D.
5. Finally: another load envelope that overlays the V—n envelope, and we take the worst cases.



McCormick

Turning performance — 1



Recall the relationships developed for turning flight:

Horizontal equilibrium $\quad L\sin\phi = m\dfrac{V^2}{R_H} = mV\omega$

Vertical equilibrium $\quad L\cos\phi = nW\cos\phi = W$

Leading to $\quad \phi = \cos^{-1}\left(\dfrac{1}{n}\right)$

$$\sin\phi = \frac{\sqrt{n^2-1}}{n}$$

From which we obtained the following for rate and radius of turn: $\quad \omega = \dfrac{g\sqrt{n^2-1}}{V} \qquad R_H = \dfrac{V^2}{g\sqrt{n^2-1}}$

Typically we wish to maximise the turn rate and minimise the turn radius. The first usually more important.

Now to consider the thrust requirement, we use the fundamental performance equation, simply (here):

$$T = D = \tfrac{1}{2}\rho V^2 S C_D = \tfrac{1}{2}\rho V^2 S(C_{D,0} + KC_L^2)$$

Now $\quad L = nW = \tfrac{1}{2}\rho V^2 S C_L \quad$ or $\quad C_L = \dfrac{2}{\rho}\dfrac{W}{S}\dfrac{n}{V^2}$

$$T = D = \tfrac{1}{2}\rho V^2 S\left[C_{D,0} + K\left(\frac{2}{\rho}\frac{W}{S}\right)^2\frac{n^2}{V^4}\right] = \frac{\rho}{2}S\left[C_{D,0}V^2 + K\left(\frac{2}{\rho}\frac{W}{S}\right)^2\frac{n^2}{V^2}\right]$$

If we set *n*=1, we recover the equation for thrust required in level flight at speed *V*. Increasing the load factor produces more induced drag at a given speed and hence demands more thrust.

Turning performance — 2

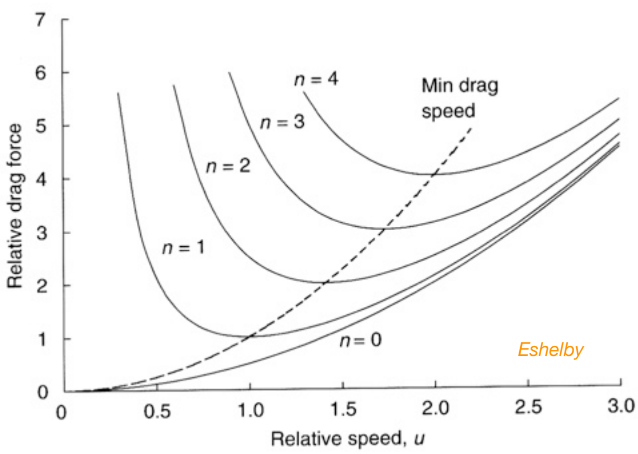
Using $\quad V = uV^* \quad$ where $\quad V^* = \left(\dfrac{2}{\rho}\dfrac{W}{S}\right)^{1/2}\left(\dfrac{K}{C_{D,0}}\right)^{1/4} \quad$ is the level-flight minimum-drag speed

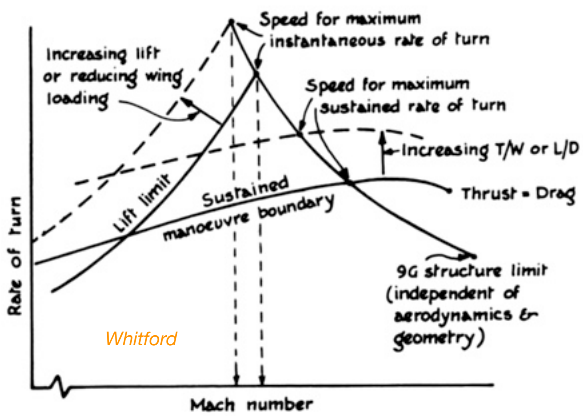
and $\quad D^* = \tfrac{1}{2}\rho V^{*2} S 2C_{D,0} \quad$ is the corresponding drag force,

we normalise the FPE: $\quad \dfrac{T}{D^*} = \dfrac{D}{D^*} = \dfrac{1}{2}\left[u^2 + \dfrac{n^2}{u^2}\right] \quad$ or $\quad \tau - \dfrac{1}{2}\left[u^2 + \dfrac{n^2}{u^2}\right] = 0$

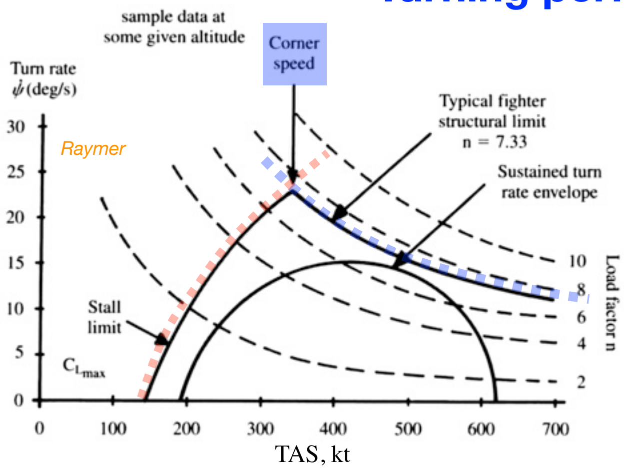
And we can see how the extra induced drag that comes with load factors *n* > 1 produces a requirement for more thrust at any speed.

We wish to find out the aircraft manoeuvre capabilities, both for instantaneous and sustained turns.





Turning performance — 3



On a rate of turn or manoeuvreability diagram one plots the sustained rate of turn vs speed that the aircraft can attain (supplied from the FPE), as well as the aerodynamic and structural limits imposed by stall and load factor.

In our analysis we'll work with a dimensionless FPE, so we'll make the rate of turn dimensionless also.

$$\omega = \frac{g\sqrt{n^2-1}}{V} = \frac{g\sqrt{n^2-1}}{uV^*} \Rightarrow \Omega = \frac{\omega V^*}{g} = \frac{\sqrt{n^2-1}}{u}$$

Structural limit just derived: $\Omega_{\text{struct}} = \dfrac{\sqrt{n_{\text{limit}}^2-1}}{u}$

Stall limit

$$V_{\text{stall}} = \sqrt{\frac{2}{\rho}\frac{W}{S}}\sqrt{\frac{n}{C_{L\,\max}}} \quad \text{and} \quad V^* = \sqrt{\frac{2}{\rho}\frac{W}{S}}\sqrt{\frac{1}{C_L^*}} \quad \text{where} \quad C_L^* = \sqrt{\frac{C_{D,0}}{K}}$$

$$\text{so} \quad u_{\text{stall}} = \frac{V_{\text{stall}}}{V^*} = \sqrt{\frac{n\,C_L^*}{C_{L\,\max}}} \quad \text{or} \quad n_{\text{stall}} = \frac{C_{L\,\max}}{C_L^*}u_{\text{stall}}^2 \Rightarrow \Omega_{\text{stall}} = \frac{\sqrt{n_{\text{stall}}^2-1}}{u} = \frac{\sqrt{u^4\left(\frac{C_{L\,\max}}{C_L^*}\right)^2-1}}{u}$$

$$\Omega_{\text{stall}} = \sqrt{u^2\left(\frac{C_{L\,\max}}{C_L^*}\right)^2 - \frac{1}{u^2}}$$

Solving $\Omega_{\text{struct}} = \Omega_{\text{stall}}$ for *u* gives the corner speed (we already had it for given $C_{L\max}$ and n_{limit}).
The aim is to develop optimal relationships.

Turning performance — 4

Now we have the structural and aerodynamic limiting values, compute the sustained turn rate on the assumption that thrust is invariant with speed.

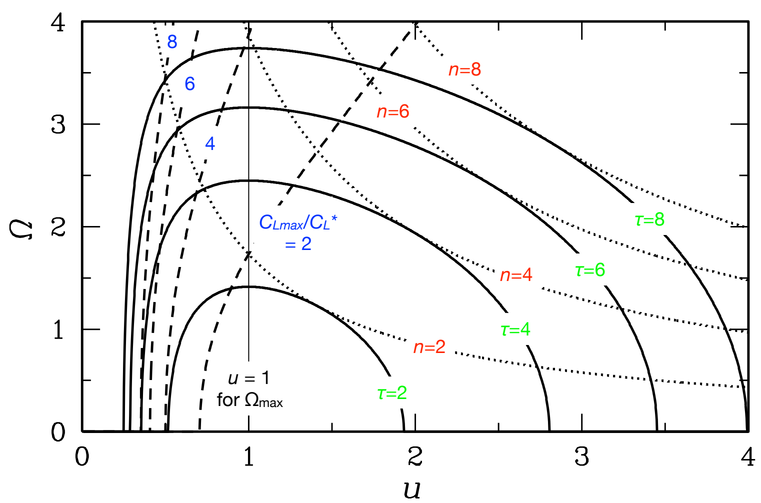
From $\tau - \dfrac{1}{2}\left[u^2 + \dfrac{n^2}{u^2}\right] = 0$ and $\Omega = \dfrac{\sqrt{n^2-1}}{u}$ i.e. $n^2 = \Omega^2u^2+1 \Rightarrow u^4 + \left(\Omega^2 - 2\tau\right)u^2 + 1 = 0$

Obtain solution to quadratic: $u_{1,2} = \left[\tau - \dfrac{\Omega^2}{2} \pm \dfrac{1}{2}\left(\Omega^4 - 4\tau\Omega^2 + 4\tau^2 - 4\right)^{1/2}\right]^{1/2}$

There is a single value of *u* (i.e. a TP) when the discriminant is zero, i.e. at max Ω, $u = \sqrt{\tau - \Omega^2/2}$

Using also the condition that the discriminant is zero, we obtain $u_{\Omega\,\max} = 1$ and $\Omega_{\max} = \sqrt{2\tau - 2}$

I.e. the maximum sustained turn rate always occurs at *u*=1. For given τ and *u*, find: $\Omega = \sqrt{2\tau - \dfrac{u^4+1}{u^2}}$



Limit type
Thrust
Stall
Structure

The instantaneous corner-speed turn rate is typically significantly larger than the sustained turn-rate capability.

Turning performance — 5

We just found the maximum sustained turn rate (MSTR) $\Omega_{\max} = \sqrt{2\tau - 2}$

By similar means one can find the conditions for a sustained minimum-radius (sharpest) turn, SST.

Or the sustained turn that produces maximum sustained load factor *n.*

Defining also a dimensionless turn radius $r = gR_H/V^{*2}$ one can tabulate the maximum dimensionless sustained-turn values permitted by the drag polar and constant thrust. (One should compare these to the limits imposed by structural strength and stall.)

Special case	u	n	Ω	r
MSTR	1	$\sqrt{2\tau - 1}$	$\sqrt{2\tau - 2}$	$\dfrac{1}{\sqrt{2\tau - 2}}$
SST	$\dfrac{1}{\sqrt{\tau}}$	$\dfrac{\sqrt{2\tau^2 - 1}}{\tau}$	$\sqrt{\dfrac{\tau^2 - 1}{\tau}}$	$\dfrac{1}{\sqrt{\tau^2 - 1}}$
$n_{\max}$	$\sqrt{\tau}$	τ	$\sqrt{\dfrac{\tau^2 - 1}{\tau}}$	$\dfrac{1}{\sqrt{\tau^2 - 1}}$

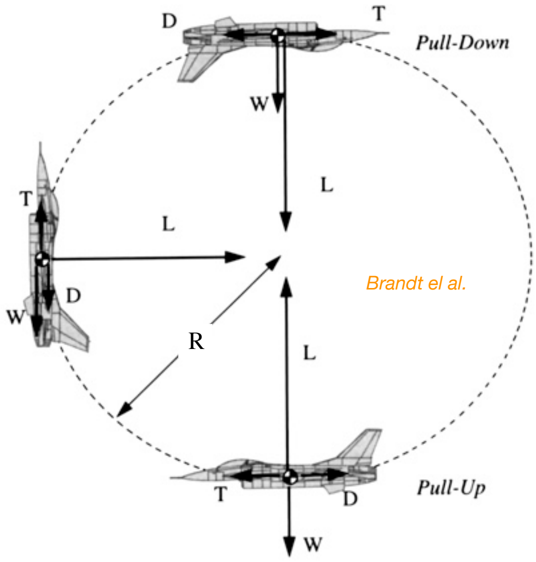
Pamadi

Note that these values are for (jet) aircraft where the thrust is assumed independent of speed.

Similar relationships could be derived for prop aircraft where the power is assumed independent of speed.

(Instantaneous) Pull-up and Pull-down

1. For instantaneous manoeuvres, we don't worry about having enough thrust to maintain airspeed. Pull-up and pull-down from level flight are typical.



2. At Pull-up $m\dfrac{V_\infty^2}{R} = L - W$

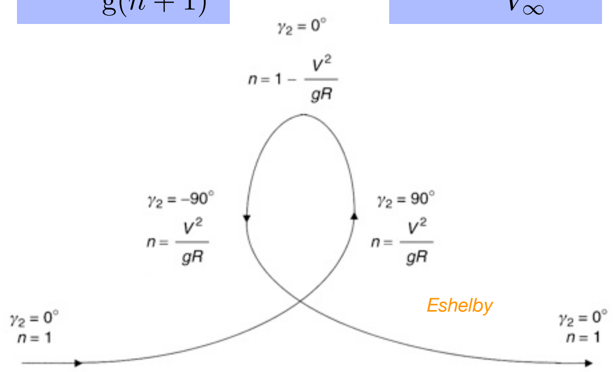
$$R = \frac{mV_\infty^2}{L - W} = \frac{W}{g}\frac{V_\infty^2}{L - W} = \frac{V_\infty^2}{g(L/W - 1)} = \frac{V_\infty^2}{g(n-1)}$$

$$\omega = \frac{V_\infty}{R} = \frac{g(n-1)}{V_\infty}$$

3. At Pull-down $m\dfrac{V_\infty^2}{R} = L + W$

$$R = \frac{V_\infty^2}{g(n+1)} \qquad \omega = \frac{g(n+1)}{V_\infty}$$

4. One consequence is that looping manoeuvres typically do not have a true circular fight path.



Limiting cases for large load factor

1. Radius R

Level turn	Pull up	Pull down	Large-*n* limit (n≫1)
$R = \dfrac{V_\infty^2}{g\sqrt{n^2-1}}$	$R = \dfrac{V_\infty^2}{g(n-1)}$	$R = \dfrac{V_\infty^2}{g(n+1)}$	$R = \dfrac{V_\infty^2}{gn}$

2. Turn rate ω

Level turn	Pull up	Pull down	Large-*n* limit (n≫1)
$\omega = \dfrac{g\sqrt{n^2-1}}{V_\infty}$	$\omega = \dfrac{g(n-1)}{V_\infty}$	$\omega = \dfrac{g(n+1)}{V_\infty}$	$\omega = \dfrac{gn}{V_\infty}$

3. Dependence on aircraft and flight parameters

$$L = \tfrac{1}{2}\rho V_\infty^2 S C_L, \quad V_\infty^2 = \frac{2L}{\rho S C_L}$$

$$R = \frac{V_\infty^2}{gn} = \frac{2L}{\rho S C_L g n} = \frac{2L}{\rho S C_L g(L/W)} = \frac{2}{\rho C_L g}\frac{W}{S}$$

$$\omega = \frac{gn}{V_\infty} = \frac{gn}{\sqrt{2L/(\rho S C_L)}} = \frac{gn}{\sqrt{[2n/(\rho C_L)](W/S)}} = g\sqrt{\frac{n\rho C_L}{2(W/S)}}$$

$$R_{\min} = \frac{2}{\rho g (C_L)_{\max}}\frac{W}{S}$$

$$\omega_{\max} = g\sqrt{\frac{\rho (C_L)_{\max} n_{\max}}{2(W/S)}}$$

$$n_{\max} = \tfrac{1}{2}\rho V_\infty^2 \frac{(C_L)_{\max}}{W/S}$$