1

Tennekes (2009) The Simple Science of Flight

Snow goose (*Anser caerulescens*).

Gossamer Albatross: $W = 940$ N, $S = 70$ m^2, $b = 29$ m.

Range and endurance fundamentals

Recommended reading:
Torenbeek & Wittenberg: Chapters 5, 6 & 9
Ruijgrok, Chapter 15.

Questair Venture: $W = 8{,}000$ N, $S = 6.76$ m^2, $b = 8.40$ m, $P = 224$ kW (300 hp).

2

Range (R) and endurance (E) in level powered flight

All range and endurance tasks for powered flight considered here rely on stored energy sources (fuel).

For generality let's say the initially stored on-board energy is of chemical-bond type, with total amount C and conversion rate $\mathrm{d}C/\mathrm{d}t = P_C$, the power available from conversion of this energy to another type.

We will use the mass-specific energy density of this storage, H (with dimensions energy per unit mass) and also its mass density ρ_f (dimensions mass per unit volume). The total amount of stored energy $C = m_f H$. Total fuel weight $W_f = g\, m_f$.

<u>Note the interesting fact that H/g has dimensions of distance. **The maximum achievable range for each kind of fuel is approximately the same order as H/g.**</u>

	mass-specific	weight-specific
Fuel	***H* (MJ/kg)**	***H/g* (km)**
Hydrogen	120	12200
Methane	50	5100
Natural gas	45	4600
Jet fuel/kerosene/petrol/diesel	42.5	4350
Fat	30	3060
Peanut butter	27	2750
Sugar	15	1530
Lithium-ion battery	0.9	92

For air-burned fuels, H is the (lower) heating value of the fuel. For many liquid hydrocarbon fuels, $H \approx 42$ MJ/kg.

As fuel is burned, its mass flow rate is $\dot{m}_f = \mathrm{d}m_f/\mathrm{d}t$ and its weight flow rate $\mathrm{d}W_f/\mathrm{d}t = \mathrm{g}\,\mathrm{d}m_f/\mathrm{d}t$.

Note that as fuel is burned, aircraft weight W decreases. $\mathrm{d}W/\mathrm{d}t = -\mathrm{d}W_f/\mathrm{d}t$. or $\mathrm{d}W = -\mathrm{d}W_f$

Whenever we convert this stored energy to another type, there is an associated conversion efficiency ($\eta \le 1$). There may be a number of conversion steps before we get to useful (e.g. drag-resisting) power. The overall efficiency to that stage is η_o.

Potentially η_o can be broken down into a number of sub-conversion efficiencies, e.g. $\eta_o = \eta_{el} \times \eta_{pr}$ (electrical, propulsive)

<u>Regardless of the particular energy system, the organising principle is: $\eta_o\, P_C$ = useful power = $T\, V$.</u>

3

Range (R) and endurance (E) in level powered flight

Stored energy is converted to useful work done, which for winged flying vehicles is usually estimated in steady level flight and is the product of thrust times distance or power times duration.

The fundamental flight mechanics equation for steady level flight is $T = D = \dfrac{W}{(L/D)}$

The fundamental thermodynamics equation for steady level flight is $\eta_o P_C = \eta_o \dfrac{\mathrm{d}C}{\mathrm{d}t} = VT = \dfrac{VW}{(L/D)}$

Considering range R and endurance E we note: $\mathrm{d}E \equiv \mathrm{d}t$ and $\dfrac{\mathrm{d}R}{\mathrm{d}t} \equiv \dfrac{\mathrm{d}x}{\mathrm{d}t} \equiv V$

Non-fuel-burning aircraft (W = const).

Endurance $\eta_o \dfrac{\mathrm{d}C}{\mathrm{d}t} = \eta_o \dfrac{\mathrm{d}C}{\mathrm{d}E} = \dfrac{VW}{(L/D)}$ or $\mathrm{d}C = \dfrac{VW}{\eta_o (L/D)}\mathrm{d}E$ or $\Delta C = \displaystyle\int_{t=t_1}^{t=t_2} \dfrac{VW}{\eta_o (L/D)}\mathrm{d}E$

For a segment where speed etc. is steady: $\dfrac{\eta_o (L/D)}{VW}\Delta C = \Delta E$

To maximise endurance for a given amount of energy and weight W, we want to fly in a way that maximises $\eta_o(L/D)/V$. Unsurprisingly, this is flight at the minimum-power airspeed. If η_o is constant, that means flying at a point on the drag polar where $C_L{}^3/C_D{}^2$ is a maximum (this occurs because $V \propto C_L{}^{-1/2}$).

If we consume all the stored energy on board at the same flight condition, then $E = \dfrac{\eta_o H (L/D)}{\mathrm{g}V} \dfrac{W_f}{W}$

Range $\eta_o \dfrac{\mathrm{d}C}{\mathrm{d}t} = \dfrac{VW}{(L/D)} = \dfrac{W}{(L/D)} \dfrac{\mathrm{d}R}{\mathrm{d}t}$ or $\eta_o \mathrm{d}C = \dfrac{W}{(L/D)}\mathrm{d}R$ or $\Delta C = \displaystyle\int_{t=t_1}^{t=t_2} \dfrac{W}{\eta_o (L/D)}\mathrm{d}R$

For a segment where speed etc. is steady: $\dfrac{\eta_o (L/D)}{W}\Delta C = \Delta R$

If we consume all the stored energy on board at the same flight condition, then $R = \dfrac{\eta_o H (L/D)}{\mathrm{g}} \dfrac{W_f}{W}$

4

Range (R) and endurance (E) of electric powered aircraft

Battery energy capacity C (J); mass-specific energy density H (J/kg), battery weight $W_f = \mathrm{g}C/H$ (N).

Electrical power $PC = \mathrm{d}C/\mathrm{d}t$.

To maximise endurance *E* for a given amount of energy *C*, minimise energy per unit time (i.e. power).

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \frac{\mathrm{d}C}{\mathrm{d}E} = P_C \qquad \mathrm{d}C = P_C\,\mathrm{d}E = \frac{TV_\infty}{\eta_E \eta_{pr}}\mathrm{d}E = \frac{WV}{\eta_E \eta_{pr} C_L/C_D}\mathrm{d}E$$

where $V = \left(\dfrac{2}{\rho}\dfrac{W}{S}\right)^{1/2} \left(\dfrac{1}{C_L}\right)^{1/2}$

Hence $\mathrm{d}C = \dfrac{W\left(\frac{2}{\rho}\frac{W}{S}\right)^{1/2}}{\eta_E \eta_{pr} C_L^{3/2}/C_D}\mathrm{d}E$ and again assuming constant operational conditions, integrate to get

$$E = \frac{\eta_E \eta_{pr}}{W\left(\frac{2}{\rho}\frac{W}{S}\right)^{1/2}} \frac{C_L^{3/2}}{C_D} C$$

We fly at the minimum power point on the drag polar – $(C_L{}^{3/2}/C_D)_{max}$ – and minimise weight, etc etc.

Range (R) and endurance (E) of electric powered aircraft

Battery energy capacity C (J); mass-specific energy density H (J/kg), battery weight $W_f = \mathrm{g}C/H$ (N).

Electrical power $P_C = \mathrm{d}C/\mathrm{d}t$.

To maximise range R for a given amount of energy C, minimise energy use per unit distance $C/\Delta x$

$$\frac{C}{\Delta x} \equiv \dot{C}\frac{\Delta t}{\Delta x} \equiv \frac{P_C}{V} \quad \text{or since} \quad V = \frac{\mathrm{d}x}{\mathrm{d}t} \equiv \frac{\mathrm{d}R}{\mathrm{d}t} \quad \text{we have} \quad \dot{C}\frac{\Delta t}{\Delta x} = \frac{\mathrm{d}C}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\mathrm{d}C}{\mathrm{d}R} = \frac{P_C}{V}$$

$$\mathrm{d}C = \frac{P_C}{V}\mathrm{d}R$$

Now $\eta_E \eta_{pr} P_C = TV$ where η_E is electric power conversion efficiency and η_{pr} is propeller efficiency.

$$P_C = \frac{TV}{\eta_E \eta_{pr}} \quad \text{or} \quad \frac{P_C}{V} = \frac{T}{\eta_E \eta_{pr}} \quad \text{so} \quad \mathrm{d}C = \frac{T}{\eta_E \eta_{pr}}\mathrm{d}R \quad \text{which NB is just} \quad \mathrm{d}C = \frac{D}{\eta_E \eta_{pr}}\mathrm{d}R$$

so that

$$R = \frac{\eta_E \eta_{pr}}{D}C$$

Maximize range by minimizing drag: fly at (L/D)* and reduce $qC_{D0}S$.

we can integrate to get

$$R = \frac{\eta_E \eta_{pr} C_L/C_D}{W}C$$

Splitting weight W into airframe and battery weight (W_e and W_C)

$$R = \frac{\eta_E \eta_{pr} C_L/C_D}{W_e + \mathrm{g}C/H}C$$

Interestingly, as airframe weight $W_e \rightarrow 0$, we have $R = \eta_E \eta_{pr} C_L/C_D \dfrac{H}{\mathrm{g}}$ independent of C.

(Recall earlier remark that maximum range is always of order H/g for any fuel type.)

Range (R) and endurance (E) in fuel-burning flight

Fuel-burning aircraft

Estimation of range and endurance has to account for the fact that fuel burn reduces aircraft weight.

$$F \doteq \mathrm{d}W_f/\mathrm{d}t = \dot{m}_f\mathrm{g} = -\mathrm{d}W/\mathrm{d}t$$ is weight flow rate of fuel.

The (weight-) *specific endurance* $1/F$ is the time airborne per unit of fuel weight consumed.

$$\frac{\mathrm{d}t}{\mathrm{d}W_f} = \frac{1}{\mathrm{d}W_f/\mathrm{d}t} = \frac{1}{F}$$

$$\mathrm{d}E \equiv \mathrm{d}t$$

The (weight-) *specific range* V/F is the distance covered during the airborne time.

$$\frac{\mathrm{d}R}{\mathrm{d}W_f} = \frac{\mathrm{d}R/\mathrm{d}t}{\mathrm{d}W_f/\mathrm{d}t} = \frac{V}{F}$$

Recall:

$$\frac{\mathrm{d}R}{\mathrm{d}t} \equiv \frac{\mathrm{d}x}{\mathrm{d}t} \equiv V$$

Say we fly from time labelled 0 to time labelled 1, then the weight of fuel consumed $W_f = W_0 - W_1$ We note that $\mathrm{d}W_f = -\mathrm{d}W$

We see that the endurance and range can be considered as integrals.

$$E = \int \mathrm{d}t = -\int_{W_0}^{W_1} \frac{\mathrm{d}W}{F} = \int_{W_1}^{W_0} \frac{\mathrm{d}W}{F} = \int_{W_1}^{W_0} \frac{\mathrm{d}W}{\dot{m}_f\mathrm{g}}$$

$$R = \int V\mathrm{d}t = -\int_{W_0}^{W_1} \frac{V}{F}\mathrm{d}W = \int_{W_1}^{W_0} \frac{V}{F}\mathrm{d}W = \int_{W_1}^{W_0} \frac{V}{\dot{m}_f\mathrm{g}}\mathrm{d}W$$

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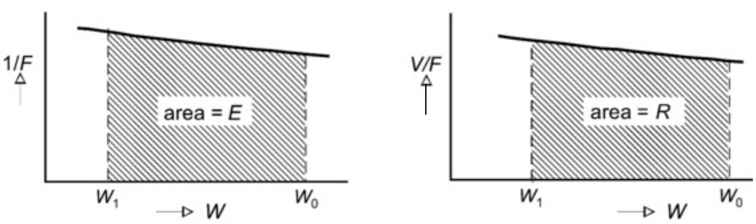


(a) Endurance (b) Range

Range (R) and endurance (E) in level fuel-burning flight

Fuel-burning aircraft

First we give a generic treatment of range and endurance that isn't specific to powerplant type.

$$\eta_o PC = \eta_o \dot{m}_f H = -\eta_o \frac{\mathrm{d}W}{\mathrm{d}t}\frac{H}{\mathrm{g}} = \frac{VW}{(L/D)}$$

Recall: $\mathrm{d}E \equiv \mathrm{d}t$ and $\dfrac{\mathrm{d}R}{\mathrm{d}t} \equiv \dfrac{\mathrm{d}x}{\mathrm{d}t} \equiv V$ or $\mathrm{d}R = V\mathrm{d}E$

Endurance

$$-\eta_o \frac{\mathrm{d}W}{\mathrm{d}t}\frac{H}{\mathrm{g}} = \frac{V}{(L/D)}\mathrm{d}E \qquad \mathrm{d}E = -\frac{\eta_o}{V}\frac{H}{\mathrm{g}}\frac{L}{D}\frac{\mathrm{d}W}{W} \qquad E = \int_{W_1}^{W_0} \frac{\eta_o}{V}\frac{H}{\mathrm{g}}\frac{L}{D}\frac{\mathrm{d}W}{W}$$

Range

$$-\eta_o \frac{\mathrm{d}W}{W}\frac{H}{\mathrm{g}} = \frac{V}{(L/D)}\mathrm{d}R \qquad \mathrm{d}R = -\eta_o \frac{H}{\mathrm{g}}\frac{L}{D}\frac{\mathrm{d}W}{W} \qquad R = \int_{W_1}^{W_0} \eta_o \frac{H}{\mathrm{g}}\frac{L}{D}\frac{\mathrm{d}W}{W}$$

If segment 0–1 (say) is flown at constant speed and angle of attack, then η_o and L/D are constant:

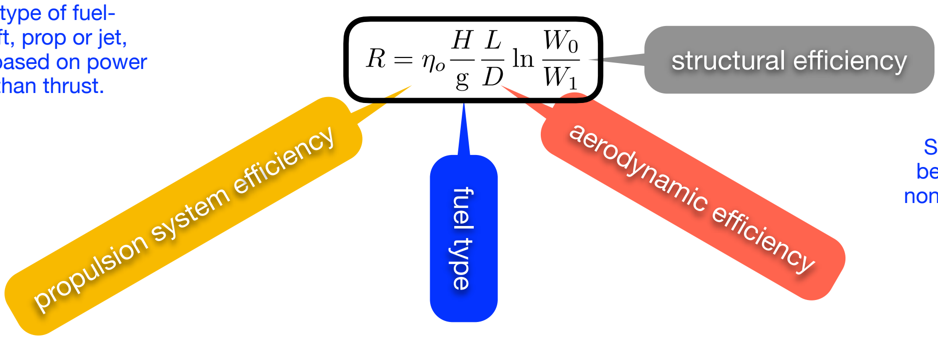
$$E = \frac{\eta_o}{V}\frac{H}{\mathrm{g}}\frac{L}{D}\ln\frac{W_0}{W_1} \qquad R = \eta_o \frac{H}{\mathrm{g}}\frac{L}{D}\ln\frac{W_0}{W_1}$$

The range equation is very interesting because it shows directly the sensitivity of R to:

- H/g, the weight-specific energy of the fuel type used;
- η_o, a figure of merit for the propulsion system;
- C_L/C_D, the aerodynamic efficiency;
- W_0/W_1, a figure of merit for the structural system (larger values indicate greater ability to carry fuel).

Range optimisation considerations

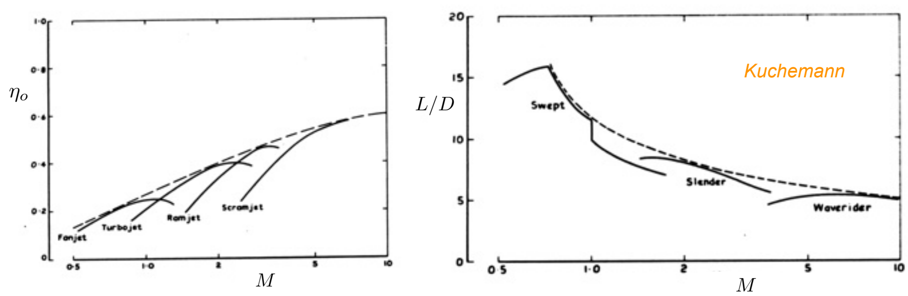
This form of range equation works for any type of fuel-burning aircraft, prop or jet, because it is based on power output rather than thrust.



Similar developments can be made for endurance, or non-fuel-burning propulsion (e.g. batteries).

This initially seems like a straightforward division of optimisation targets but in reality, aerodynamics and structures are coupled through the weight ratio W_0/W_1.

Note also that there is an interesting effect where overall propulsive efficiency tends to rise even while L/D falls with increasing Mach number. This implies that perhaps much the same range can be achieved with the same value of W_0/W_1 across a wide range of speeds.



Range (R) and endurance (E) in level fuel-burning flight

Propeller- and jet-type fuel-burning aircraft

One usually deals somewhat differently with powerplants rated on shaft power P_S (typically, propeller-powered aircraft, where propulsive efficiency is broken out as a separate item) or on thrust T (typically, jet-powered aircraft, where it is not).

For propeller-powered aircraft, $P_S \doteq \dfrac{\dot{m}_f}{c_p}$ (NB: different from specific excess power P_s.)

where c_p is the power-specific mass flow rate of fuel (dimensions: [mass/(power × time)])

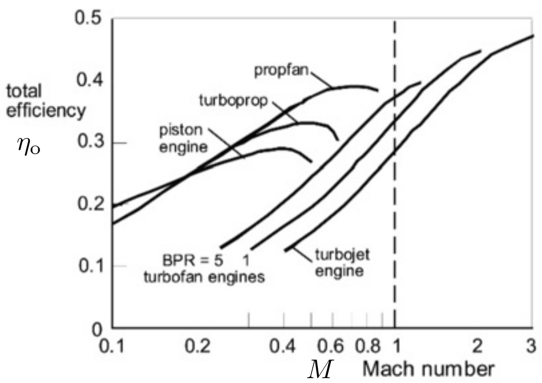
and the total efficiency is split up as $\eta_o = \eta_{pr} \times \eta_{th}$ = propulsive efficiency × thermal efficiency

From $\eta_o P_C = \eta_{pr} P_S \; (= TV)$ one finds $\eta_{tot} = \eta_{pr}/(c_p H)$

For jet-powered aircraft, $T \doteq \dfrac{\dot{m}_f}{c_t}$

where c_t is the thrust-specific mass flow rate of fuel (dimensions: [mass/(thrust × time)]).

From $\eta_o P_C = TV$ one finds $\eta_o = V/(c_t H) = Ma/(c_t H)$



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Range (R) and endurance (E) in level fuel-burning flight

Breguet-type relations for fuel-burning aircraft

Starting from the power- or thrust-specific fuel consumption rate definitions, the fundamental thermodynamic relation for steady level flight $\eta_o P_C = \dfrac{VW}{(L/D)}$ and using $\dot{m}_f = \dfrac{1}{g}\dfrac{dW_f}{dt} = -\dfrac{1}{g}\dfrac{dW}{dt}$

we derive equations for the two different aircraft classes

	Prop	**Jet**	
	$-\eta_{pr}\dfrac{1}{gc_p}\dfrac{dW}{dt} = \dfrac{VW}{L/D}$	$-\dfrac{V}{gc_t}\dfrac{dW}{dt} = \dfrac{VW}{L/D}$	**From fundamental equation**
	rearrange to get		
$dE \equiv dt$	$-\dfrac{\eta_{pr}(L/D)}{gc_p V}\dfrac{dW}{W} = dE$	$-\dfrac{1}{gc_t}\dfrac{C_L}{C_D}\dfrac{dW}{W} = dE$	**Endurance**
$\dfrac{dR}{dt} \equiv \dfrac{dx}{dt} \equiv V$	$-\dfrac{\eta_{pr}(L/D)}{gc_p}\dfrac{dW}{W} = dR$	$-\dfrac{V}{gc_t}\dfrac{C_L}{C_D}\dfrac{dW}{W} = dR$	**Range**

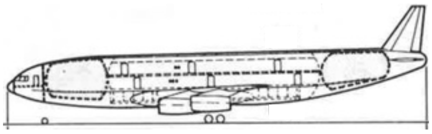
To integrate the equations in analytical form, assumptions have to be made about flight protocols.

The best range for jet aircraft can be obtained if speed and angle of attack (hence V, c_t, C_L/C_D) are all constant and the aircraft climbs continuously as the weight reduces.

In which case $R = \dfrac{V}{gc_t}\dfrac{C_L}{C_D}\ln\dfrac{W_0}{W_1}$ c.f., previously: $R = \eta_o\dfrac{H}{g}\dfrac{C_L}{C_D}\ln\dfrac{W_0}{W_1}$ (the same statement)

This is the standard form for the Breguet range equation.

LH_2 – liquid hydrogen fuel



1. Liquid hydrogen has a higher energy density per unit mass but a lower energy density per unit volume than conventional hydrocarbon fuels (see table). TSFC: roughly 1/3 (i.e. 42/120) of CH_X.
2. LH_2 is a cryogen – only helium has a lower boiling point. This has some unusual design consequences.
3. Those two facts imply some significant changes are required in aircraft sizing compared to hydrocarbon fuels. For the same range requirement and compared to an equivalent CH_X aircraft:
    a. wings of LH_2 fuelled aircraft are smaller (fuel weight is lower);
    b. fuel volume is larger than for CH_X and the (pressurised, insulated, more heavy) tanks typically need to be more roughly spherical in shape – these features imply reduced values of $(L/D)_{max}$.
4. H_2 can either be oxidised in gas turbine engines (higher efficiency is possible than for CH_X) or by using fuel cells to convert to electricity, driving electric motors. Both require significant technical development. Gas turbines seem the more likely immediate choice for long-range passenger aircraft.

	Hydrogen[7]	Methane[7]	Jet A[8,*]
Nominal composition	H_2	CH_4	$CH_{1.93}$
Molecular weight	2.016	16.04	168[9]
Heat of combustion (low)			
kJ/g	120	50.0	42.8
(Btu/lb)	(51,590)	(21,500)	(18,400)
Liquid density			
g/cm³ at 283 K	0.071[b]	0.423[b]	0.811
(lb/ft³)	(4.43)	(26.4)	(50.6)
Specific heat[b]			
J/g-K	9.69	3.50	1.98[10]
(Btu/lb-°F)	(2.32)	(0.84)	(0.47)
Boiling point at 1 atm			
K	20.27	112	440—539
(°F)	(−423)	(−258)	(332—510)
Freezing point			
K	14.4	91	233
(°F)	(−434)	(−296)	(−41)
Heat of vaporization at 1 atm			
J/g	446	510	360[10]
(Btu/lb)	(192)	(219)	(155)

Property	Effect (relative to jet fuel)
High heat of combustion	Fuel weight reduced by factor of 2.8 Quieter aircraft
High specific heat	Fuel cools engine and vehicle hot parts High TIT and OPR Further reduced SFC Further weight saving
Low density	Lesser weight of fuel requires about 4.15 × more volume; this leads to Lower L/D Low wing loading at takeoff
Cryogenic	Requires Airtight insulation system Heavy tank and fuel system Special tank fill and vent procedures Constant tank pressure to minimize boil-off

See Brewer (1991), *Hydrogen Aircraft Technology*, CRC Press.

TSFC and total efficiency

TSFC (c_t) can be supplied either on a mass-specific basis (as we have done above) or on a weight-specific basis.

In SI units, the mass-specific values are in kg/(N.s) – but it is customary to use kg/(N.hr).

In SI units, the weight-specific values are in N/(N.s) – but it is customary to use N/(N.hr).
Either of these differ from (are less than) the mass-specific values by a factor of g.

In Imperial units (as appear in many texts and in industry), TSFC values are usually given in lb/(lbf.hr).

Note that the numerical values of pound mass (lb) and of pound force (lbf) are the same.

Consequently, the weight-specific TSFC in SI units is numerically equal to the mass-specific TSFC in Imperial units.

I.e. we can use numerical c_t values we find supplied in Imperial units directly in SI calculations, provided we keep in mind they are to be considered as weight-specific TSFC.

$$\eta_o \doteq \frac{\text{thrust power developed by engine}}{\text{rate of fuel energy added to the engine}} = \frac{TV}{\dot{m}_f H} = \frac{TMa}{\dot{m}_f H} = \frac{Ma}{c_t H} \quad \text{where} \quad a = a_{sl}\sqrt{\theta}$$

θ = relative atmospheric temperature

When c_t is given on a mass-specific basis, $\eta_o = \dfrac{a_{sl}}{H}\dfrac{M\sqrt{\theta}}{c_t}$ NB: $c_t \geq (a_{sl}M\sqrt{\theta})/H$

When c_t is given on a weight-specific basis, $\eta_o = \dfrac{a_{sl}}{H/\mathrm{g}}\dfrac{M\sqrt{\theta}}{c_t}$ NB: $c_t \geq (a_{sl}M\sqrt{\theta})/(H/\mathrm{g})$

E.g. for jet fuel, c_t in lb/(lbf.hr) we have $\eta_o = \dfrac{340.3 \times 3600}{4350 \times 10^3}\dfrac{M\sqrt{\theta}}{c_t} = 0.2816\dfrac{M\sqrt{\theta}}{c_t}$ and $c_t \geq 0.2816\,M\sqrt{\theta}$

Note that for similar engine types, cruise speeds and altitudes, total efficiency will be much the same (and c_t will vary with fuel type according to ratio of H).

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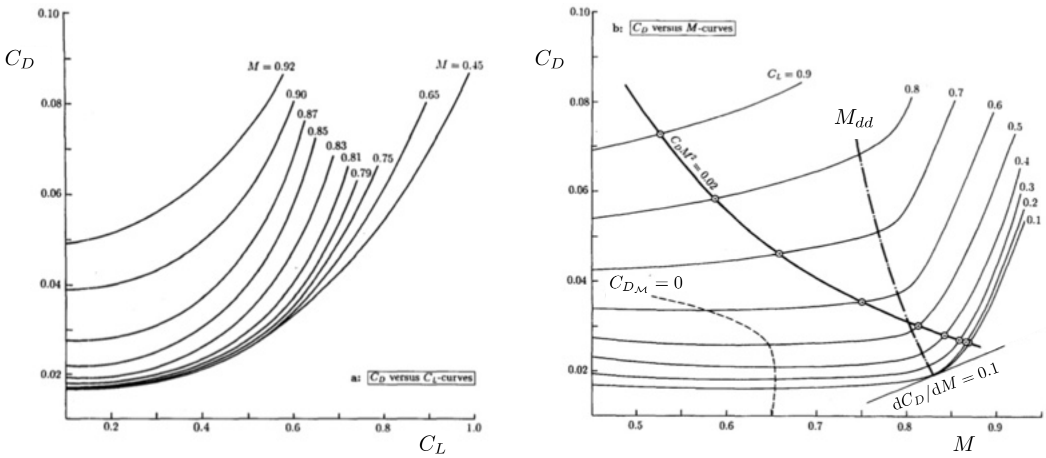
Optimal range for transonic jet aircraft

See Torenbeek *J Aircraft* V20 (1983), Prog Aerosp Sci V33 (1997), Advanced Aircraft Design (2013).

$$R = \eta_o \frac{H}{\mathrm{g}}\frac{C_L}{C_D}\ln\frac{W_0}{W_1} = \frac{V}{\mathrm{g}c_t}\frac{C_L}{C_D}\ln\frac{W_0}{W_1} = \frac{Ma}{\mathrm{g}c_t}\frac{C_L}{C_D}\ln\frac{W_0}{W_1} = \frac{a_0}{\mathrm{g}c_t/\sqrt{\theta}}M\frac{C_L}{C_D}\ln\frac{W_0}{W_1}$$

<u>Optimal aerodynamic efficiency C_L/C_D</u>

The initial focus here is on the term C_L/C_D. However, $C_D = C_D(C_L, M)$ in the transonic regime and above.



A common definition of the drag-divergence Mach number is
$M_{dd} = M|_{\mathrm{d}C_D/\mathrm{d}M=0.1}$

As shown here this is a function of C_L.

To make progress we consider (a) partial optima and (b) logarithmic differentiation.

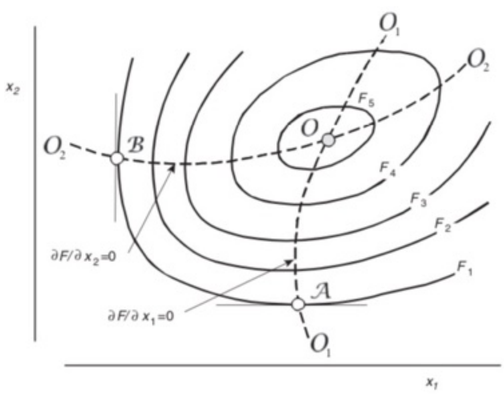
14

<u>Partial Optima</u>

An optimum for a two-dimensional function *F* occurs where the partial optima $\partial F/\partial x_n = 0$ intersect.

At an optimum, the differential

$$\mathrm{d}F = \frac{\partial F}{\partial x_1}\mathrm{d}x_1 + \frac{\partial F}{\partial x_2}\mathrm{d}x_2 = 0$$



<u>Logarithmic differentiation</u>

$$\frac{\mathrm{d}}{\mathrm{d}x}\log f(x) = \frac{\mathrm{d}f(x)/\mathrm{d}x}{f(x)} = \frac{f'(x)}{f(x)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\log(fg) = \frac{\mathrm{d}}{\mathrm{d}x}\log f + \frac{\mathrm{d}}{\mathrm{d}x}\log g = \frac{f'}{f} + \frac{g'}{g}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\log\frac{f}{g} = \frac{\mathrm{d}}{\mathrm{d}x}\log f - \frac{\mathrm{d}}{\mathrm{d}x}\log g = \frac{f'}{f} - \frac{g'}{g}$$

So the optimum for C_L/C_D occurs where $\mathrm{d}\log(C_L/C_D) = \mathrm{d}\log C_L - \mathrm{d}\log C_D = 0$

but $C_D = C_D(C_L, M)$, hence $\mathrm{d}\log C_D = \dfrac{\partial \log C_D}{\partial \log C_L}\mathrm{d}\log C_L + \dfrac{\partial \log C_D}{\partial \log M}\mathrm{d}\log M \equiv C_{D_{\mathcal{L}}}\mathrm{d}\log C_L + C_{D_{\mathcal{M}}}\mathrm{d}\log M$

where the logarithmic partial derivatives are given shorthand notations:

$$C_{D_{\mathcal{L}}} \doteq \frac{\partial \log C_D}{\partial \log C_L} = \frac{C_L}{C_D}\frac{\partial C_D}{\partial C_L} \quad (\text{constant } M)$$

$$C_{D_{\mathcal{M}}} \doteq \frac{\partial \log C_D}{\partial \log M} = \frac{M}{C_D}\frac{\partial C_D}{\partial M} \quad (\text{constant } C_L)$$

These can be interpreted as a percentage change in a dependent variable divided by a given percentage change of the independent variable.

so that $\mathrm{d}\log(C_L/C_D) = \mathrm{d}\log C_L - \mathrm{d}\log C_D = 0$ becomes $(1 - C_{D_\mathcal{L}})\,\mathrm{d}\log C_L - C_{D_\mathcal{M}}\,\mathrm{d}\log M = 0$

Clearly we need $C_{D_\mathcal{L}} = 1$ and $C_{D_\mathcal{M}} = 0$ at the optimum for C_L/C_D., i.e.

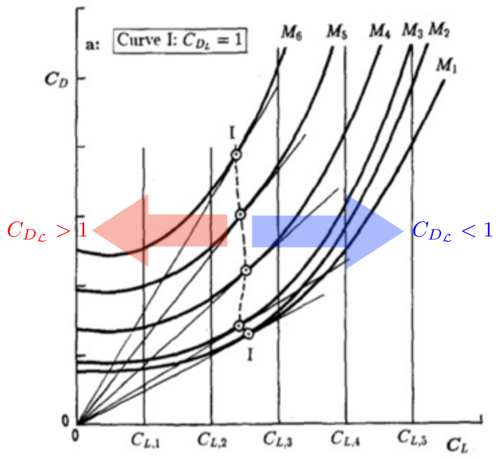
$$C_{D_\mathcal{L}} = 1 \;\rightarrow\; \frac{\partial C_D}{\partial C_L} = \frac{C_D}{C_L} \quad (\text{constant } M)$$

$$C_{D_\mathcal{M}} = 0 \;\rightarrow\; \frac{\partial C_D}{\partial M} = 0 \quad (\text{constant } C_L)$$

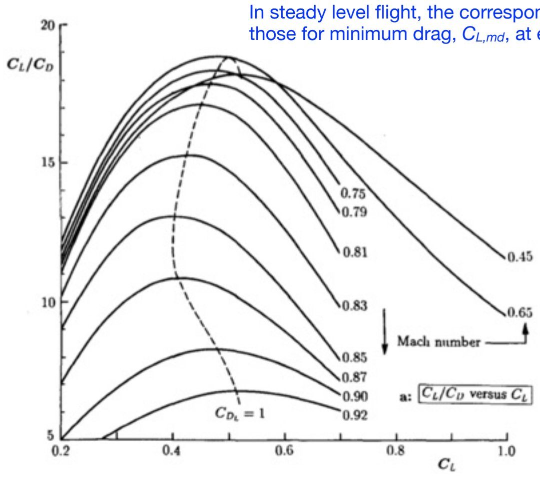
This is true before compressibility effects produce an increase in C_D with M and is a common assumption in many texts.

$C_{D_\mathcal{L}} = 1$ corresponds to the locus I on a C_D vs C_L plot:

and, since clearly C_L/C_D takes a local maximum at those locations, with the maxima on a C_L/C_D vs C_L plot:



In steady level flight, the corresponding C_L are those for minimum drag, $C_{L,md}$, at each M.



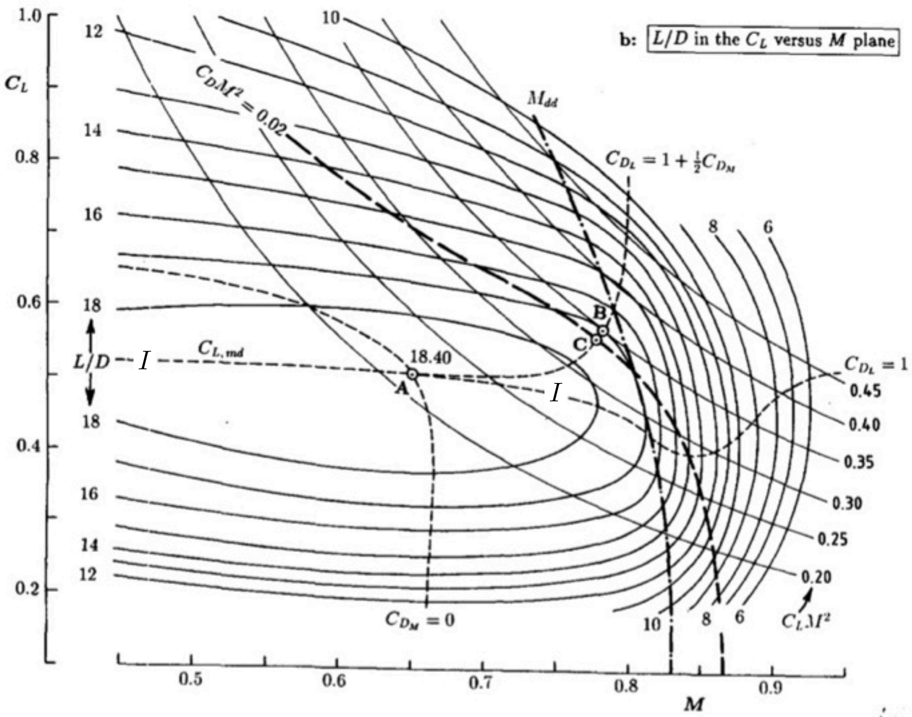
Note that peak aerodynamic efficiency occurs in the transonic regime. This happens because the linearised compressibility effect on C_L

$$C_L \approx \frac{C_{L,0}}{\sqrt{1 - M^2}}$$

initially starts to push C_L/C_D up before the nonlinear compressibility effect elevating C_D overcomes it.

$C_{D_\mathcal{M}} = 0$ forms the lower boundary of the Mach number regime where compressibility starts to degrade C_D at any C_L (see curve $C_{D_\mathcal{M}} = 0$ on earlier C_D vs M plot).

Next, consider the (M, C_L) map of C_L/C_D . If no constraints are applied we can see the global optimum occurs where the partial optima $C_{D_\mathcal{L}} = 1$ and $C_{D_\mathcal{M}} = 0$ intersect at point *A* where *L/D** = 18.40.



A constraint can move us away from the unconstrained optimum. For example, a requirement to fly at a given altitude with L=W gives

$$C_L M^2 = \frac{2W/S}{\gamma p} = \text{constant}$$

or

$$\mathrm{d}\log C_L + 2\,\mathrm{d}\log M = 0$$

Substitution into

$$(1 - C_{D_\mathcal{L}})\,\mathrm{d}\log C_L - C_{D_\mathcal{M}}\,\mathrm{d}\log M = 0$$

gives

$$C_{D_\mathcal{L}} = 1 + \tfrac{1}{2} C_{D_\mathcal{M}}$$

Since above the onset of compressibility effects,

$$C_{D_\mathcal{M}} > 0 \implies C_{D_\mathcal{M}} > 1$$

(and hence $C_L > C_L$*), the constrained maxima for C_L/C_D must lie along points of tangency between C_L/C_D = const and $C_L M^2$ = const, as shown.

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Optimal value of parameter MC_L/C_D

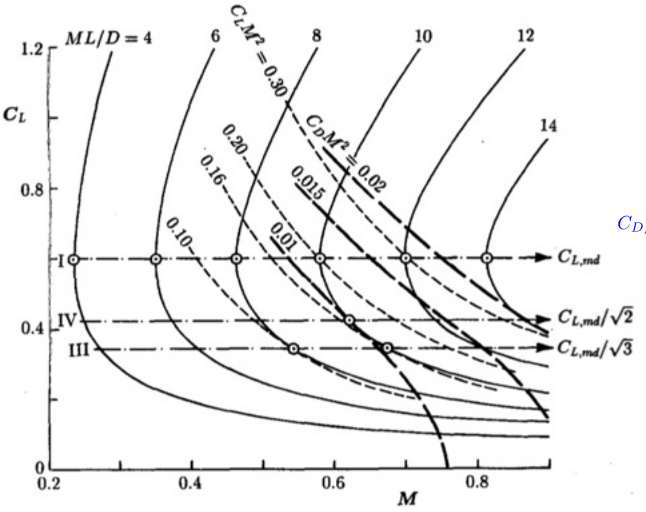
If TSFC c_t is independent of M, then the parameter to be optimised for range is MC_L/C_D. Again using logarithmic differentiation we have

$$\mathrm{d}\log M + \mathrm{d}\log C_L - \mathrm{d}\log C_D = 0$$

and since $\mathrm{d}\log C_D = C_{D_\mathcal{L}}\mathrm{d}\log C_L + C_{D_\mathcal{M}}\mathrm{d}\log M$ we obtain $(1 - C_{D_\mathcal{L}})\,\mathrm{d}\log C_L + (1 - C_{D_\mathcal{M}})\,\mathrm{d}\log M = 0$

First, examine what happens if there is no compressibility effect on C_D, i.e. $C_{D_\mathcal{M}} = 0$

(That is a common assumption in many simplified texts.) In that case, for a simple quadratic drag polar, the contours of ML/D look like this:



Clearly there is no unconstrained maximum.

At a given C_L, L/D is independent of M and so ML/D increases without bound with increasing M. Best range would be obtained at $C_L = C_L^* = C_{L,md}$. For a given W and with C_L fixed at this value, altitude has to also increase indefinitely with increasing M.

If altitude is fixed, then C_LM^2 = const. After some manipulation, one finds that $C_{D_\mathcal{L}} = 1/2$ and finally that this occurs at a maximum of C_L/C_D^2, a well-known result leading to $C_L = C_L^*/\sqrt{3}$ and $V = 3^{1/4}V^*$. This occurs along the locus of tangencies between ML/D = const. and C_LM^2 = const., as shown.

Alternatively, if thrust is fixed, one finds that C_DM^2 = const, and eventually that this occurs for maximum C_L^2/C_D^3, at which $C_L = C_L^*/\sqrt{2}$ and $V = 2^{1/4}V^*$, another well-known result (see e.g. Nicolai & Carichner). This occurs along the locus of tangencies between ML/D = const. and C_DM^2 = const., as shown.

In this fixed-thrust case, one additionally obtains

$$C_L = \sqrt{\frac{C_{D,0}}{2K}} \qquad L/D = \frac{2^{3/2}}{3}(L/D)^* = 0.9428(L/D)^* \qquad V = 2^{1/4}V^* = 1.189V^*$$

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Now return to optimising ML/D in the presence of compressibility effects, and with

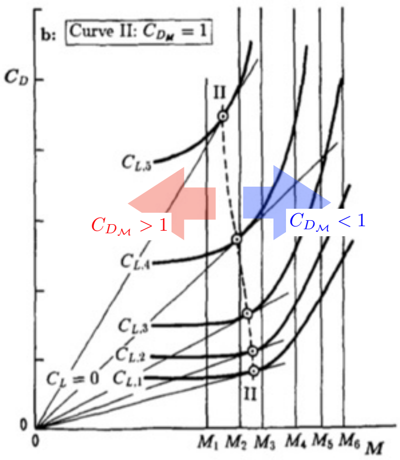
$$(1 - C_{D_\mathcal{L}})\,\mathrm{d}\log C_L + (1 - C_{D_\mathcal{M}})\,\mathrm{d}\log M = 0$$

We already dealt with finding the locus of partial optimum $C_{D_\mathcal{L}} = 1$ from a plot of C_D vs C_L, leading to curve I.

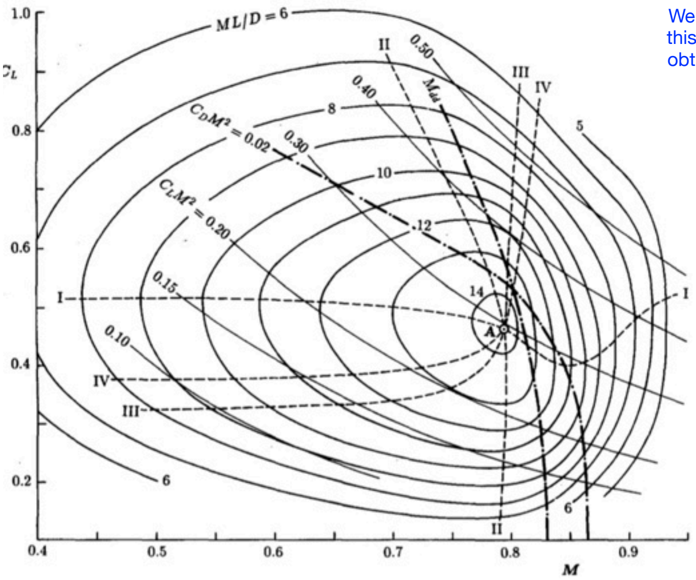
Similarly, we can find where

$$C_{D_\mathcal{M}} = 1$$

from a C_D vs M plot, leading to another partial optimum locus, Curve II:



The contour plot for MC_L/C_D is obtained by multiplying the plot for C_L/C_D by M, with the two partial optima I & II intersecting at point A, with M just slightly lower than M_{dd}. L/D is 17.61, 95.7% of (maximum+compressible) L/D^* = 18.40 previously obtained. The maximum of ML/D = 14.0 occurs at M = 0.795.



We don't actually need to make this contour plot. We could just obtain the partial optima curves I and II from drag polars (as shown earlier) and find their intersection on the C_L vs M plane.

An obvious problem in the initial design office phase is how to model the polars in the presence of compressibility without resorting to CFD or a wind tunnel.

Curves III and IV correspond to constraints for constant altitude (C_LM^2 = const) and constant thrust (C_DM^2 = const). See texts.

Optimal value of parameter $\eta_o C_L/C_D$

Recall $R = \eta_o \frac{H}{g}\frac{C_L}{C_D}\ln\frac{W_0}{W_1}$ or, for brevity, $= \eta\frac{H}{g}\frac{C_L}{C_D}\ln\frac{W_0}{W_1}$

More generally TSFC c_t is *not* independent of M, so that the parameter to be optimised for range is $\eta C_L/C_D$.

This requires $\mathrm{d}\log\eta + \mathrm{d}\log C_L - \mathrm{d}\log C_D = 0$

Proceeding similarly to before, one obtains $(1 - C_{D_\mathcal{L}})\,\mathrm{d}\log C_L + (\eta_\mathcal{M} - C_{D_\mathcal{M}})\,\mathrm{d}\log M = 0$

where $\eta_\mathcal{M} \doteq \frac{\mathrm{d}\log\eta}{\mathrm{d}\log M} = \frac{M}{\eta}\frac{\mathrm{d}\eta}{\mathrm{d}M}$