Aircraft performance

Torenbeek & Wittenberg Ch 6

Brandt et al. Ch 5

Anderson Ch 6

Aircraft performance summary

Aircraft performance considered here involves only simple concepts, mechanics, and maths.

Key inputs

- Drag = $\mathcal{F}$ (Lift): drag polar model
- Aircraft maximum lift coefficient C_{Lmax}
- Powerplant (propeller and jet) models
- Standard Atmosphere model

Key ideas

Energy (power) supplied = energy (power) dissipated (in steady state flight)

Thrust = drag (in steady state flight)

Lift = n × weight (load factor n = 1 typical)

TOPICS CONSIDERED

Equations of motion

Fundamental performance equation

Aircraft energy state

Steady level flight

Minimum thrust or power required to fly

Altitude envelope: maximum sustainable altitude

Speed envelope: minimum and maximum speeds

Fuel use: Range and endurance (air-breathing)

Steady climbing flight

Maximum rate or angle of climb

Gliding (steady descending) flight

Range or endurance for given altitude loss

Steady turning flight

Level banked turns: turn speed and radius

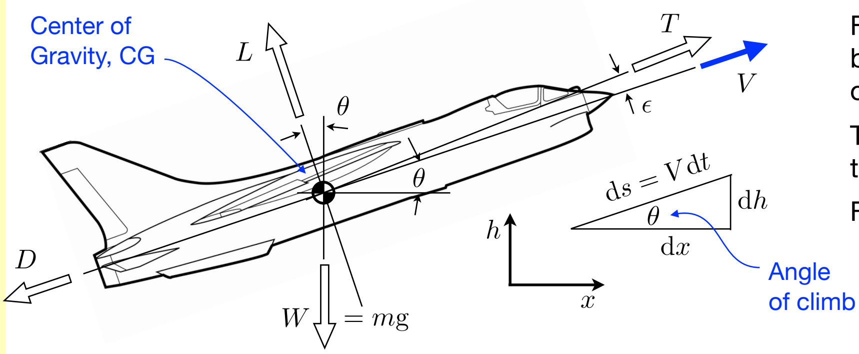
Drag increase in turning flight

Turning in the vertical plane

Take-off and landing

Runway distance requirements

Aircraft equations of motion



First consider flight where there is no bank angle, and all forces and motion occur in the aircraft plane of symmetry.

The aircraft is assumed to be *trimmed* so that moments about CG sum to zero.

From flight path geometry we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = V\cos\theta$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \dot{h} = V\sin\theta$$

Newton's second law is $\sum \boldsymbol{F} = \dfrac{\mathrm{d}(m\boldsymbol{V})}{\mathrm{d}t} = m\dfrac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} + \boldsymbol{V}\dfrac{\mathrm{d}m}{\mathrm{d}t} = m\dot{\boldsymbol{V}} + \boldsymbol{V}\dot{m} \approx m\dot{\boldsymbol{V}}$

Note that it is usual in aircraft performance dynamics to ignore the time rate of change of mass, except when computing fuel consumption. This assumption may be inadequate when fuel burn rates are high, e.g. for performance analysis of missiles.

Now we consider components tangential and normal to the flight path, leading to

Tangential $\quad m\dfrac{\mathrm{d}V}{\mathrm{d}t} = T\cos\epsilon - D - mg\sin\theta = m\dot{V}$

Normal $\quad mV\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = T\sin\epsilon + L - mg\cos\theta = mV\dot{\theta} \equiv m\dfrac{V^2}{R_V}$ where R_V is flight path radius of curvature.

Rearrange:

(Divide tangential equation through by *mg* = *W*.)

$$\frac{\dot{V}}{g} = \frac{T\cos\epsilon - D}{W} - \sin\theta$$

(Divide normal equation through by *mgV*.)

$$\frac{\dot{\theta}}{g} = \frac{T\sin\epsilon + L}{VW} - \frac{\cos\theta}{V}$$

Aircraft equations of motion

Since the thrust and lift are functions of altitude and speed, the drag is additionally a function of lift, and the fuel burn rate depends on thrust, we have a set of five ODEs to consider, four of which are coupled:

Integrate to get range $\quad \dot{x} = V\cos\theta \quad$ (not directly coupled to the remaining four)

Integrate to get *h* $\quad \dot{h} = V\sin\theta$

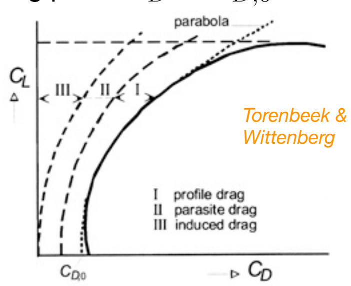
Integrate to get *W* $\quad \dot{W} = -gc_t(h,V)T(h,V)$

Integrate to get *V* $\quad \dfrac{\dot{V}}{g} = \dfrac{T(h,V)\cos\epsilon - D(h,V,L)}{W} - \sin\theta$

Integrate to get *θ* $\quad \dfrac{\dot{\theta}}{g} = \dfrac{T(h,V)\sin\epsilon + L(h,V)}{VW} - \dfrac{\cos\theta}{V}$

Note coupling of terms between equations: cannot solve them independently

Drag polar $C_D = C_{D,0} + KC_L^2$



The standard approach for most performance analysis is to obtain decoupling by assuming that θ=const., and base everything around the 4th equation in the set. i.e. the one containing the drag polar.

(Also, since ϵ is typically small or zero, cosϵ→1 and sinϵ→ϵ.) Thus, starting with the 4th equation,

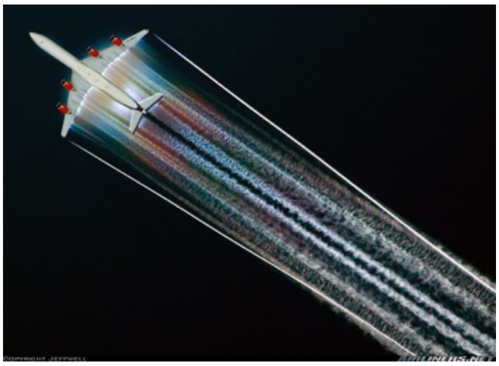
$$\frac{(T-D)V}{W} = V\sin\theta + V\frac{\dot{V}}{g} = \frac{\mathrm{d}}{\mathrm{d}t}\left(h + \frac{V^2}{2g}\right) = \frac{\mathrm{d}e}{\mathrm{d}t}$$

Fundamental Performance Equation (FPE)

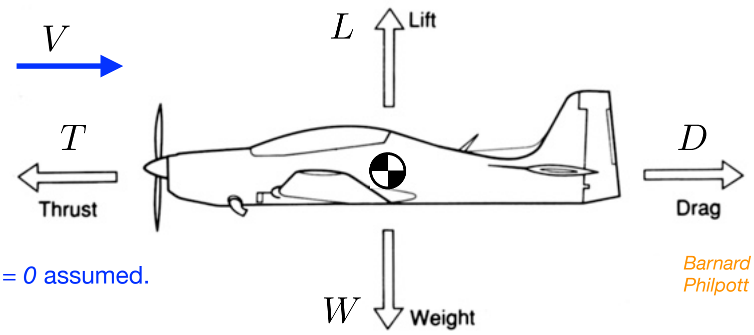
where *e*, the sum of potential and kinetic energies per unit weight, is called the aircraft's specific energy or energy height. The equation is often called the Fundamental Performance Equation and is the basis for most performance analysis (and design for performance). The rate of change of KE is often ignored.

The term (*T*–*D*)/*W* is called the specific excess thrust and (*T*–*D*)*V*/*W* is called the specific excess power, i.e. the amount of thrust/power per unit weight available to increase the aircraft's altitude or speed, or both.

Level unaccelerated flight



Analysis of steady level flight



$\epsilon = 0$ assumed.

Barnard & Philpott

(Some artistic liberty is used in this diagram: typically L is MUCH greater than *D*.)

Note that $L/D = W/T$.

(And so, correspondingly, W is typically much greater than T.)

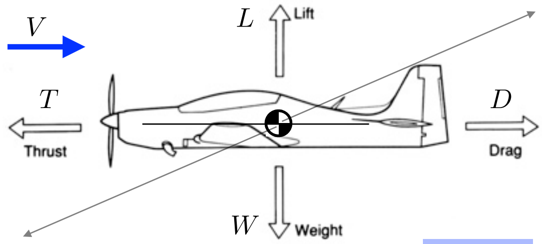
Understanding steady level flight is the key to understanding aircraft performance.

Central to this understanding is the relationship between lift, drag, speed and altitude (air density).

Questions we want to obtain answers to:

1. How are drag (thrust) and speed related?
2. How are power and speed related?
3. What is the effect of altitude on these relationships?
4. How fast and high can an aircraft fly?
5. How far can an aircraft travel with a given amount of fuel?
6. How long can an aircraft stay in the air with a given amount of fuel?

Lift, drag & thrust in level flight



Returning to
$$\frac{\dot V}{g} = \frac{T\cos\epsilon - D}{W} - \sin\theta$$
$$\frac{\dot\theta}{g} = \frac{T\sin\epsilon + L}{VW} - \frac{\cos\theta}{V}$$

with $\dot V = 0,\quad \theta = 0,\quad \dot\theta = 0$ and $\epsilon = 0$

We have the simple relationships $T = D$, $W = L$

From similar triangles
$$\frac{T}{W} = \frac{D}{L} = \frac{C_D}{C_L} = \frac{1}{C_L/C_D}$$

This is the fundamental performance equation in its most simplified form: $\dfrac{T-D}{W} = 0$

Thrust required to fly: $T_R = \dfrac{W}{C_L/C_D}$ Minimum thrust required to fly: $T_{R,\min} = \dfrac{W}{(C_L/C_D)_{\max}}$

We now look at the relationship between thrust required (drag) and speed, given *W*, *S* and altitude (*ρ*).

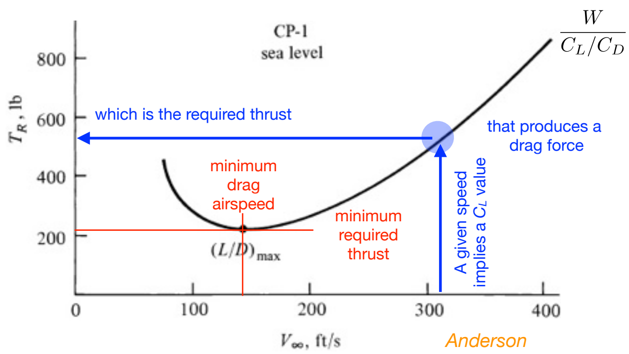
❶ $L = \frac{1}{2}\rho V^2 S C_L = W$

❷ $C_L = \dfrac{1}{\frac{1}{2}\rho V^2}\dfrac{W}{S} = \dfrac{2}{\rho}\dfrac{W}{S}\dfrac{1}{V^2}$

❸ $C_D = C_{D,0} + KC_L^2$

❹ $\dfrac{C_L}{C_D} = \dfrac{C_L}{C_{D,0} + KC_L^2}$

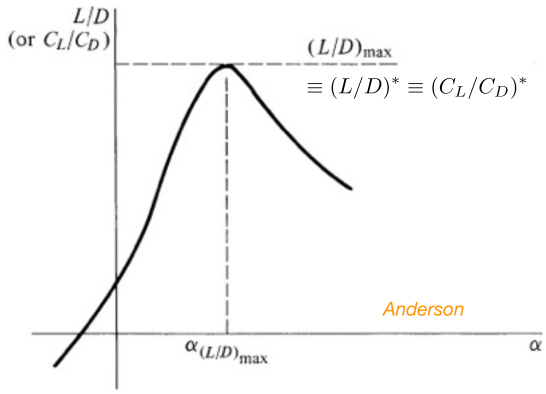
❺ $T = D = \dfrac{W}{C_L/C_D}$

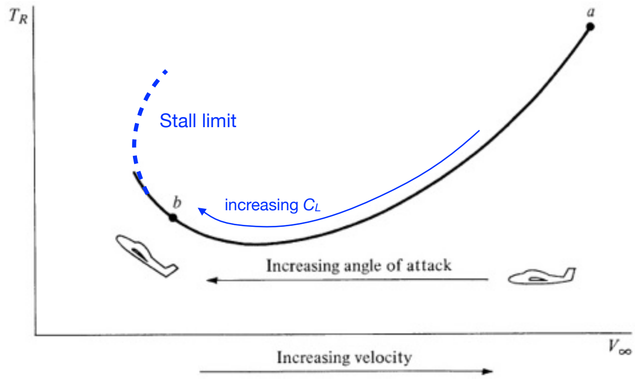


We find there is a definite speed corresponding to minimum drag in level flight and for larger thrusts there could be two possible speeds.

Lift, drag & thrust in level flight

The underlying reason is that the lift/drag ratio depends on angle of attack and C_L through the speed of flight.





Through some analysis

$$D = q_\infty S C_D = q_\infty S\left(C_{D,0} + KC_L^2\right)$$

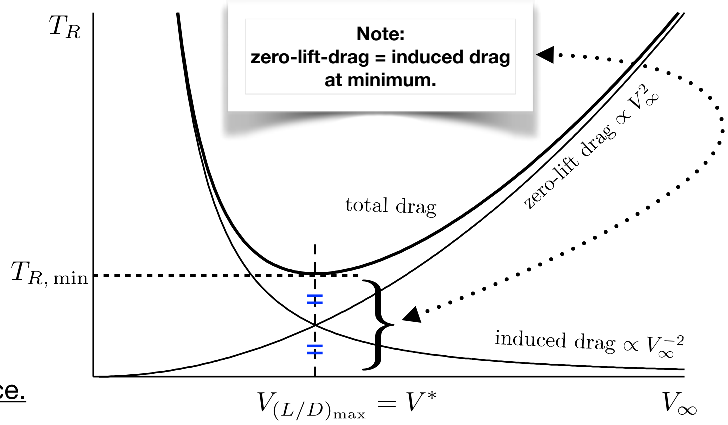
$$C_L = \frac{W}{q_\infty S}$$

$$T_R = D = S\left[\tfrac{1}{2}\rho V_\infty^2 C_{D,0} + \frac{K}{\frac{1}{2}\rho V_\infty^2}\left(\frac{W}{S}\right)^2\right]$$

zero-lift drag   lift-induced drag

we see that one contribution to drag varies with V^2, the other with V^{-2}.

Understanding and manipulating these relationships is the key to aircraft performance.



Lift, drag & thrust in level flight

Coefficient of lift at L/D_{max}, $C_L{}^*$

$$\frac{C_D}{C_L} = \frac{C_{D,0} + KC_L^2}{C_L} = \frac{C_{D,0}}{C_L} + KC_L \qquad \text{Find TP:} \quad \frac{\mathrm{d}(C_D/C_L)}{\mathrm{d}C_L} = -\frac{C_{D,0}}{C_L^2} + K = 0$$

i.e. $K = \dfrac{C_{D,0}}{C_L^{*2}}$

$$C_L^* = \sqrt{\frac{C_{D,0}}{K}}$$

$$C_D^* = 2C_{D,0}$$

Corresponding L/D_{max}, a.k.a. $(L/D)^*$, same as $(C_L/C_D)^*$

$$\left(\frac{C_L}{C_D}\right)^* = \frac{C_L^*}{C_{D,0} + KC_L^{*2}} = \frac{C_L^*}{C_{D,0} + K(C_{D,0}/K)} = \frac{C_L^*}{2C_{D,0}} = \frac{\sqrt{C_{D,0}}}{\sqrt{K}2C_{D,0}} = \frac{1}{\sqrt{4C_{D,0}K}}$$

Minimum-drag airspeed in level flight, V^*

$$V^* = \sqrt{\frac{2}{\rho}\frac{W}{S}\frac{1}{C_L^*}} = \left(\frac{2}{\rho}\frac{W}{S}\right)^{1/2}\left(\frac{K}{C_{D,0}}\right)^{1/4}$$

Dimensionless airspeed, $u = V / V^*$

$$V_\infty \equiv V = uV^* \quad \text{so} \quad C_L = \frac{2}{\rho}\frac{W}{S}\frac{1}{V^{*2}}\frac{1}{u^2} = C_L^*\frac{1}{u^2}$$

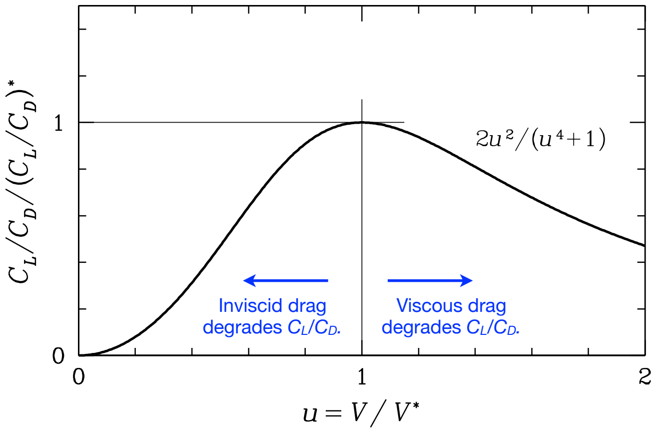
Dependence of L/D on airspeed

$$\frac{C_L}{C_D} = \frac{C_L^*/u^2}{C_{D,0} + KC_L^{*2}/u^4} = \frac{C_L^*(1/u^2)}{C_{D,0}\,(1 + 1/u^4)}$$

$$\frac{C_L}{C_D} = \frac{2}{2\sqrt{C_{D,0}K}}\frac{u^2}{u^4+1} = \left(\frac{C_L}{C_D}\right)^*\frac{2u^2}{u^4+1}$$

(dimensionless) constant × function(speed)

Recall $T_R = \dfrac{W}{C_L/C_D}$ We are gaining a clearer understanding of how T_R depends on speed.



Speed for a given thrust

What is the speed for a propulsive thrust $T_A > T_{min}$?

We see that there are two possible solutions.

$$\frac{T_A}{W} = \frac{D}{W} = \frac{D}{L} = \left(\frac{C_D}{C_L}\right)^*\frac{u^2+u^{-2}}{2} = \left(\frac{C_D}{C_L}\right)^*\frac{u^4+1}{2u^2}$$

Rearrange:

$$u^4 - 2\frac{T_A}{W}\left(\frac{C_L}{C_D}\right)^* u^2 + 1 = 0$$

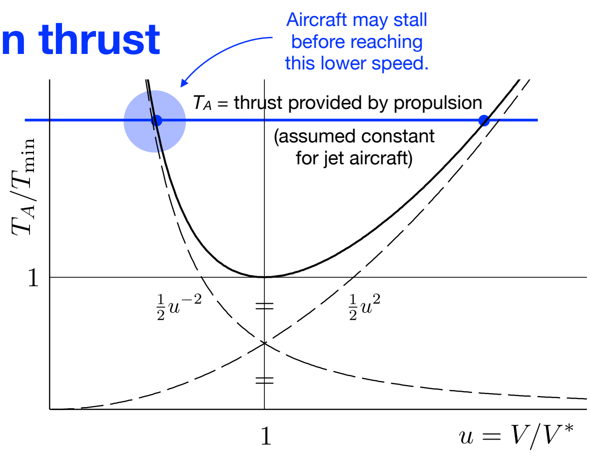
$$x^2 + px + q = 0$$

$$x = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

Solve quadratic in u^2:

$$u^2 = \frac{T_A}{W}\left(\frac{C_L}{C_D}\right)^* \pm \left(\left[\frac{T_A}{W}\left(\frac{C_L}{C_D}\right)^*\right]^2 - 1\right)^{1/2}$$

$$u_{1,2} = \left\{\frac{T_A}{W}\left(\frac{C_L}{C_D}\right)^* \pm \left(\left[\frac{T_A}{W}\left(\frac{C_L}{C_D}\right)^*\right]^2 - 1\right)^{1/2}\right\}^{1/2}$$



One can show that $u_1 u_2 = 1$

Note $V_{1,2} = V^* u_{1,2}$

TAS $$V = V^* u = \left(\frac{2}{\rho}\frac{W}{S}\frac{1}{C_L^*}\right)^{1/2} u$$

IAS $$V_e = V_e^* u = \sqrt{\sigma}V^* u$$ where $$\sigma = \frac{\rho}{\rho_{SL}}$$

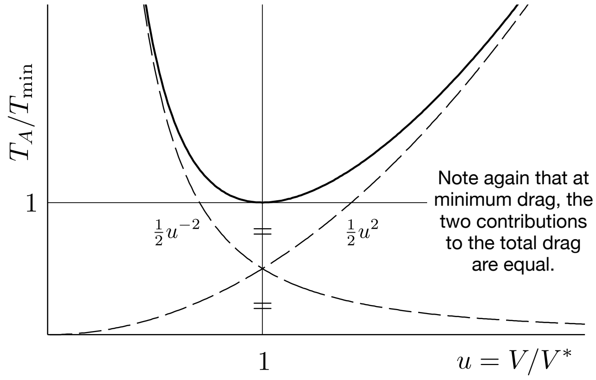
Alternatively, the equivalent without exploiting non-dimensionalization w.r.t. the minimum-drag values is:

$$V_{1,2} = \left\{\frac{(T_A/W)(W/S) \pm (W/S)\left([T_A/W]^2 - 4C_{D,0}K\right)^{1/2}}{\rho\, C_{D,0}}\right\}^{1/2}$$

Altitude effect on minimum thrust required

How does a change in altitude affect the thrust required?

$$\left(\frac{T}{W}\right)_{\min} = \left(\frac{D}{L}\right)^* = 2\sqrt{C_{D,0}K}, \qquad T_{\min} = 2W\sqrt{C_{D,0}K}$$



Note that this does not depend on altitude.
Reducing W, $C_{D,0}$ or K reduces $T_{\min}$.

We see that for this condition, the zero-lift drag equals the induced drag:

$$C_D^* = C_{D,0} + KC_L^{*2} = C_{D,0} + KC_{D,0}/K = 2C_{D,0}$$

How does a change in altitude affect the speed at minimum thrust condition?

IAS

$$V_e^* = \left(\frac{2}{\rho_0}\frac{W}{S}\right)^{1/2}\left(\frac{1}{C_L^*}\right)^{1/2} = \left(\frac{2}{\rho_0}\frac{W}{S}\right)^{1/2}\left(\frac{K}{C_{D,0}}\right)^{1/4}$$

(a constant, SL-equivalent value).

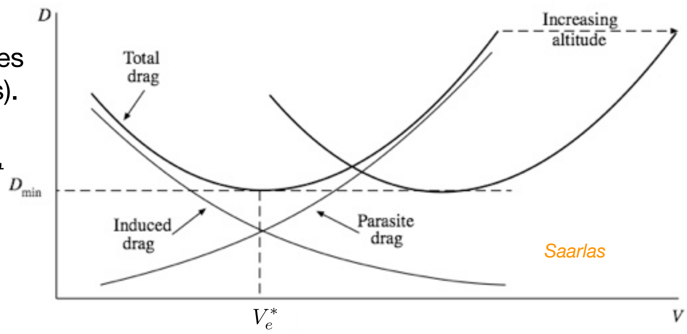
TAS

$$V^* = \left(\frac{2}{\rho}\frac{W}{S}\right)^{1/2}\left(\frac{1}{C_L^*}\right)^{1/2} = \frac{V_e^*}{\sqrt{\sigma}}$$

(recall σ reduces as h increases).

★ The minimum drag force is independent of altitude, but the corresponding TAS increases with altitude.

Provided there is enough thrust, we can fly faster by increasing altitude (or increasing wing loading).



Altitude effect on thrust available

What is the maximum altitude in level steady flight with a jet engine?

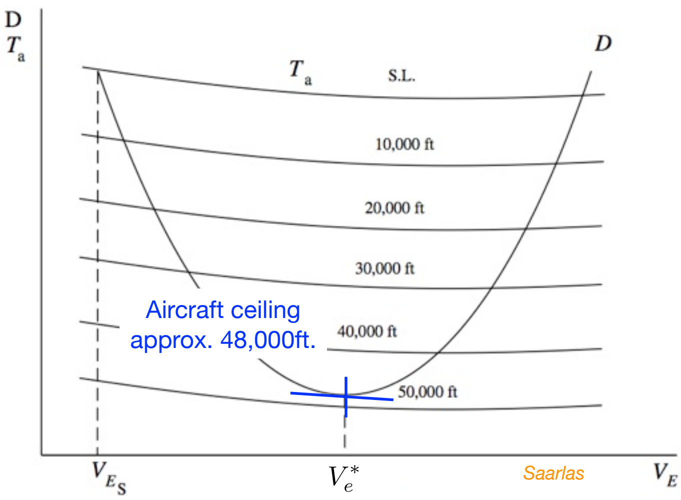
While the thrust required T_R does not change with altitude, the thrust available T_A falls.

It is most convenient to plot the drag curve with V_e (or u) as abscissa, since it is the same at all altitudes.

Over this we plot the available powerplant thrust T_A as a function of V_e and h.

The altitude at which the available thrust falls below the minimum drag is the aircraft's *ceiling*.

The approach here is graphical but if we can describe the variation of thrust with speed and altitude then we can solve the problem numerically. If we assume thrust is independent of speed and just depends on altitude, the solution becomes reasonably simple.



Recall:

A simple and reasonable approximation for cruise is

$$T_A \approx r\,T_0\left(\frac{\rho}{\rho_0}\right)^s$$

$$P_A \approx r\,P_0\left(\frac{\rho}{\rho_0}\right)^s$$

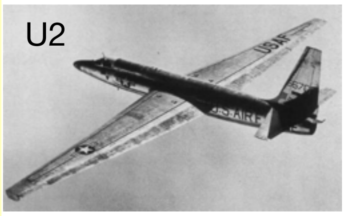
where 0 denotes sea-level static rated values and

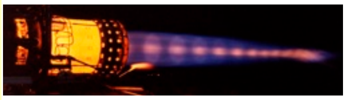
$$r \simeq 0.5, \quad s \simeq 0.7$$

I.e. thrust is only a function of h.

(The thrust model shown in the figure here is slightly more sophisticated.)

Johnson & Colman





J58 jet engine, night test

Lockheed SR71 initial design

Kelly Johnson's initial design for the SR71, a supersonic successor to the U2 spyplane, started with the choice of two J58 turbojets for propulsion.

The U2, essentially a glider with one turbofan, had an airframe mass of 2.6 t.

SR71 cruise range approx. 6500 km, speed Mach 3 at 90000 ft (approx 800 m/s).

At the design cruise altitude and speed of 90,000 ft and Mach 3, the J58 was capable of T = 18 kN thrust with TSFC c_t = 68 mg/s/N. Mass 2.7 t (each).

Preliminary aerodynamic estimates gave best L/D approx. 7.5.

So maximum initial cruise weight

$$W = T\left(\frac{C_L}{C_D}\right)^* = 2 \times 18 \times 10^3 \times 7.5 \text{ N} = 270 \text{ kN}$$

or a mass of 27.5 t.

As a very rough estimate, the amount of fuel required

$$W_f \approx \mathrm{g} \times \dot{m}_f \times \frac{\text{range}}{\text{speed}} = \mathrm{g}c_t T\frac{R}{V}$$

$$= 9.8 \times 68 \times 10^{-6} \times 2 \times 18 \times 10^3 \times \frac{6500 \times 10^3}{800} \text{ N} = 195 \text{ kN}$$

i.e. mass 20 t.

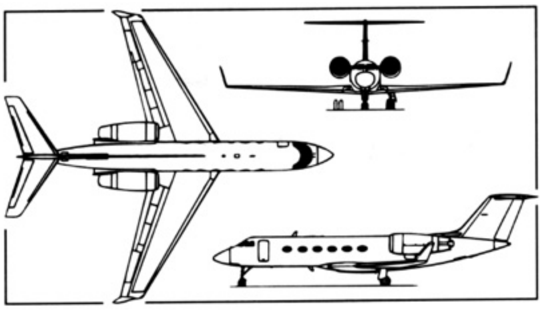
That left a target airframe mass of 27.5 – 20 – 2 × 2.7 = 2.1 t

or only 80% of the U2's airframe mass, seemingly unlikely.

In turn that left Johnson and his team with some more thinking to do!

Full story in *From Rainbow to Gusto* by Paul Suhler (AIAA).

Example – Grumman Gulfstream IV business jet



Supplied data:

W = 324 kN, S = 88.3 m², b = 23.7 m

Estimated $C_{D,0}$ = 0.015, aircraft efficiency factor e = 0.85

Engines 2x R–R turbofan, SL static thrust T_0 = 65 kN each

Thrust model $\frac{T_A}{T_0} = 0.5\left(\frac{\rho}{\rho_{SL}}\right)^{0.7} = 0.5\sigma^{0.7}$

Find

1. Minimum thrust required for level flight
2. Equivalent (SL) airspeed at this thrust (IAS)
3. Estimated minimum, maximum SL airspeeds
4. Estimated minimum, maximum airspeeds at 11km altitude
5. Estimated maximum altitude for level flight

Aerodynamic parameters

$$A = \frac{b^2}{S} = \frac{23.7^2}{88.3} = 6.36 \qquad K = \frac{1}{\pi Ae} = \frac{1}{\pi \times 6.36 \times 0.85} = 0.0589$$

$$C_L^* = \sqrt{\frac{C_{D,0}}{K}} = \sqrt{\frac{0.015}{0.0589}} = 0.505 \qquad \left(\frac{C_L}{C_D}\right)^* = \frac{1}{\sqrt{4C_{D,0}K}} = \frac{1}{\sqrt{4 \times 0.015 \times 0.0589}} = 16.8$$

1. Minimum thrust required for level flight

$$T_{R,\min} = \frac{W}{(C_L/C_D)^*} = \frac{324}{16.8}\,\text{kN} = 19.3\,\text{kN} \qquad \frac{T_{R,\min}}{W} = \frac{1}{(C_L/C_D)^*} = \frac{1}{16.8} = 0.0595$$

Example – Grumman Gulfstream IV business jet

2. Equivalent (SL) airspeed at this thrust

$$V_e^* = \sqrt{\frac{2}{\rho_0}\frac{W}{S}\frac{1}{C_L^*}} = \sqrt{\frac{2}{1.225} \times 3670 \times \frac{1}{0.505}}\,\mathrm{m/s} = 108.9\,\mathrm{m/s}$$ (Mach number M = 108.9/340.3=0.320)

3. Estimated maximum, minimum SL airspeeds

$$\frac{T_A}{W} = \frac{0.5\sigma^{0.7}T_0}{W} = \frac{0.5T_0}{W} = \frac{0.5 \times 130}{324} = 0.201 \quad (> 0.0595\,\checkmark)$$

$$\begin{aligned} u_{1,2} &= \left\{ \frac{T_A}{W}\left(\frac{C_L}{C_D}\right)^* \pm \left(\left[\frac{T_A}{W}\left(\frac{C_L}{C_D}\right)^*\right]^2 - 1 \right)^{1/2} \right\}^{1/2} \\ &= \left\{ 0.201 \times 16.8 \pm \left([0.201 \times 16.8]^2 - 1 \right)^{1/2} \right\}^{1/2} \\ &= \{3.370 \pm 3.2186\}^{1/2} = 2.567,\ 0.389 \qquad (u_1 \times u_2 = 1\,\checkmark) \end{aligned}$$

$$V_{1,2} = u_{1,2}V_e^* = \{2.567,\ 0.389\} \times 108.9\,\mathrm{m/s} = 279.6\,\mathrm{m/s},\ 42.4\,\mathrm{m/s}$$

We note that the minimum value is unrealistically low as it implies that $C_L = \frac{2}{\rho}\frac{W}{S}\frac{1}{V_2^2} = \frac{2}{1.225} \times \frac{3670}{42.4^2} = 3.33$

This could not be achieved without the aid of significant high-lift devices.

However the upper speed, 279.6m/s, corresponds to a Mach number of 279.6/340.3 = 0.822, which is not unrealistic for this type of aircraft.

AIRCRAFT PERFORMANCE

Example – Grumman Gulfstream IV business jet

4. Estimated maximum, minimum airspeeds at h_G=11km At 11km, ρ = 0.2971 × 1.225 kg/m³ (σ = 0.2971).

$$T_A = T_0 0.5\sigma^{0.7} = 130 \times 0.5 \times 0.2971^{0.7}\,\mathrm{kN} = 27.8\,\mathrm{kN} \qquad \frac{T_A}{W} = \frac{27.8}{324} = 0.0858 \quad (> 0.0595\,\checkmark)$$

$$\begin{aligned} u_{1,2} &= \left\{ \frac{T_A}{W}\left(\frac{C_L}{C_D}\right)^* \pm \left(\left[\frac{T_A}{W}\left(\frac{C_L}{C_D}\right)^*\right]^2 - 1 \right)^{1/2} \right\}^{1/2} \\ &= \left\{ 0.0858 \times 16.8 \pm \left([0.0858 \times 16.8]^2 - 1 \right)^{1/2} \right\}^{1/2} \\ &= \{1.4414 \pm 1.0381\}^{1/2} = 1.575,\ 0.635 \qquad (u_1 \times u_2 = 1\,\checkmark) \end{aligned}$$

$$V_{1,2} = u_{1,2}V^* \qquad V^* = \frac{V_e^*}{\sqrt{\sigma}} = \frac{108.9}{\sqrt{0.2971}}\,\mathrm{m/s} = 199.8\,\mathrm{m/s} \qquad V_{1,2} = 314.7\,\mathrm{m/s}, \quad 126.9\,\mathrm{m/s}$$

The lower speed implies C_L = 1.25, which is achievable without flaps. ✔

The higher speed implies M = 1.07, which is somewhat unlikely (indicates a problem with our modelling).

5. Estimated maximum altitude for level flight

Maximum altitude capability occurs where $T_A = \frac{W}{(C_L/C_D)^*} = T_{R,\min} = 19.7\,\mathrm{kN}$

$$T_A = T_0\,0.5\sigma^{0.7} = 130 \times 0.5\sigma^{0.7} = 19.7\,\mathrm{kN}$$

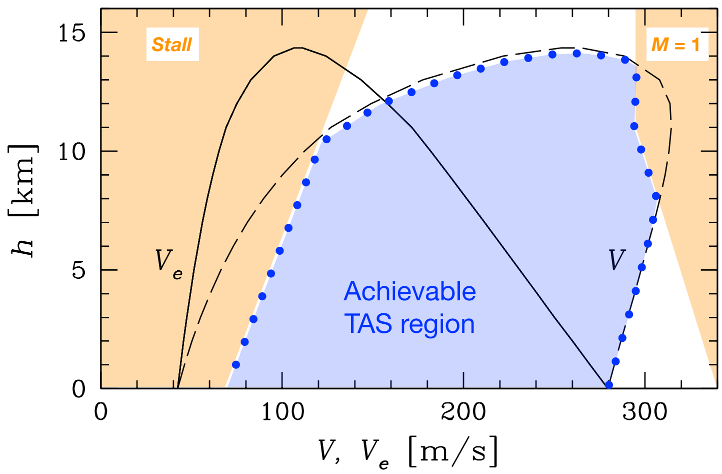
Rearrange: $\sigma = \left(\frac{19.3}{0.5 \times 130}\right)^{1/0.7} = 0.1765 \qquad \rho = \sigma\rho_0 = 0.1765 \times 1.225\,\mathrm{kg/m^3} = 0.2162\,\mathrm{kg/m^3}$

Interpolating in ISA tables, we find h = 14.35 km.

AIRCRAFT PERFORMANCE

Example – Grumman Gulfstream IV business jet

Finally we show the envelopes of equivalent and true air speeds predicted from the analysis.



Recall that

$$V = \frac{V_e}{\sqrt{\sigma}} = \frac{V_e}{\sqrt{\rho/\rho_0}}$$

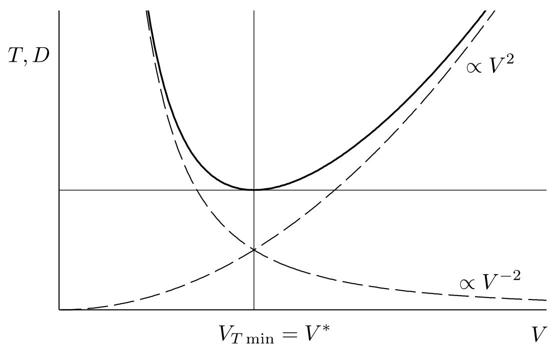
Naturally, we can fly at speeds (either EAS or TAS) inside the envelopes by reducing the throttle setting and/or changing the aircraft trim.

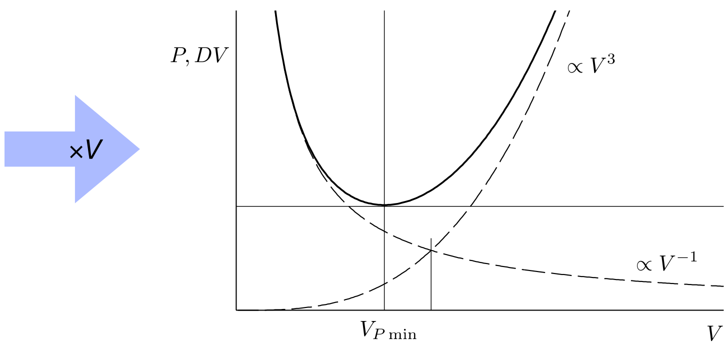
We note that other limitations may over-ride the predictions made on the basis of the propulsion model and drag polar. At the low-speed end the aircraft may stall, severely increasing drag above the simple model provided by the drag polar. At the high-speed end, Mach number (compressibility) effects may increase the drag well above that predicted by the drag polar, too.

Power in level flight

If engine performance is better characterised by available power than thrust and e.g. we are interested in minimum power required to fly, the analysis is similar but the outcomes somewhat different.

The aircraft thrust-speed curve is converted to a power-speed curve by multiplying through by velocity, since $P = TV = DV$.





We see that the minimum-power speed is lower than the minimum-thrust speed.

Thrust required to fly: $T_R = \dfrac{W}{C_L/C_D}$ Power required to fly: $P_R = T_R V_\infty = \dfrac{W}{C_L/C_D} V_\infty$

Since $V_\infty = \sqrt{\dfrac{2}{\rho}\dfrac{W}{S}\dfrac{1}{C_L}}$ we see that $P_R = \dfrac{W}{C_L/C_D}\sqrt{\dfrac{2}{\rho}\dfrac{W}{S}\dfrac{1}{C_L}}$ or $P_R = \sqrt{\dfrac{2}{\rho}\dfrac{W^3}{S}\dfrac{C_D^2}{C_L^3}}$

$$P_R = \sqrt{\frac{2}{\rho}\frac{W^3}{S}\frac{C_D^2}{C_L^3}} \propto \frac{1}{C_L^{3/2}/C_D}$$

We can see that P_R is a minimum at $(C_L^{3/2}/C_D)_{max}$ (and also at ρ_{SL}).

AIRCRAFT PERFORMANCE

Power in level flight

To find the speed for minimum power, where we have $(C_L^{3/2}/C_D)_{max}$, we return to the relationship

$$\frac{C_L}{C_D} = \left(\frac{C_L}{C_D}\right)^* \frac{2}{u^2+u^{-2}} = \left(\frac{C_L}{C_D}\right)^* \frac{2u^2}{u^4+1}$$

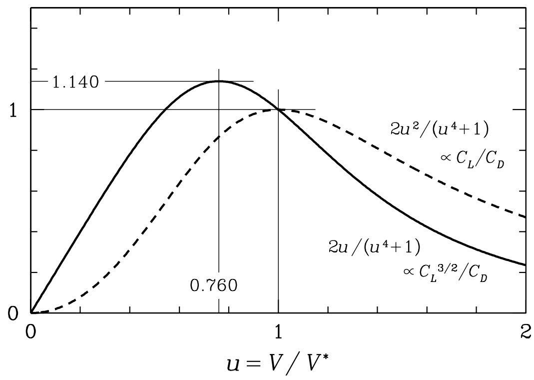
where $u = V/V^*$ $\quad \left(\frac{C_L}{C_D}\right)^* = \frac{1}{\sqrt{4C_{D,0}K}}$

$$V^* = \sqrt{\frac{2}{\rho}\frac{W}{S}\frac{1}{C_L^*}} \qquad C_L^* = \sqrt{\frac{C_{D,0}}{K}}$$

Since $W = L = \frac{1}{2}\rho V^2 S C_L = \frac{1}{2}\rho V^{*2} S C_L^*$ we have $C_L V^2 = C_L^* V^{*2}$ or $(C_L)^{1/2} = (C_L^*)^{1/2}/u$

So $\frac{C_L^{3/2}}{C_D} = C_L^{1/2}\frac{C_L}{C_D} = (C_L^*)^{1/2}\left(\frac{C_L}{C_D}\right)^* \frac{2u}{u^4+1}$

$\frac{2u}{u^4+1}$ has a maximum where $u = (1/3)^{1/4} = 0.760$



This means the airspeed for minimum power is lower than that for minimum drag by factor of $(1/3)^{1/4} = 0.760$.

If we substitute back the relationships above into

$$\frac{P_R}{W} = \sqrt{\frac{2}{\rho}\frac{W}{S}\frac{C_D^2}{C_L^3}}$$

we have the following relationship for minimum power loading, *P*/*W*:

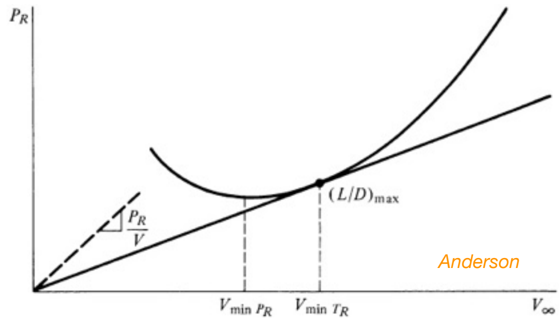
$$\left(\frac{P}{W}\right)_{min} = \left(\frac{256}{27}\right)^{1/4}\left(\frac{2}{\rho}\frac{W}{S}\right)^{1/2}(C_{D,0}K^3)^{1/4} = 1.755\left(\frac{2}{\rho}\frac{W}{S}\right)^{1/2}(C_{D,0}K^3)^{1/4}$$

Best to fly at minimum altitude, largest ρ.

More important to reduce *K* than $C_{D,0}$.

AIRCRAFT PERFORMANCE

Power in level flight



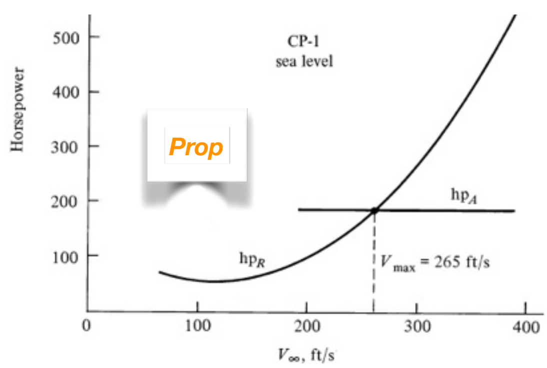
We can also easily recover the minimum-drag airspeed from the P_R vs *V* curve.

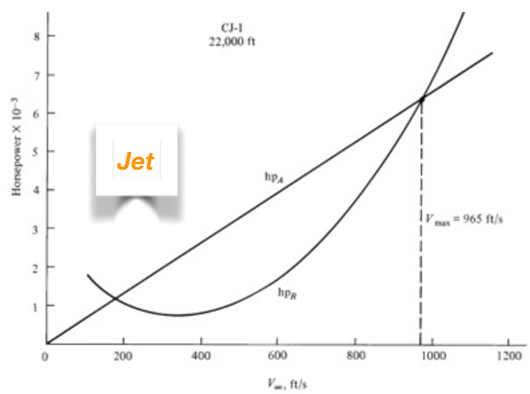
Draw a line from the origin that is just tangential to the P_R vs *V* curve. This has the minimum value of P_R/V.

But since $P_R = T_R V$, (i.e. $P_R/V = T_R$), this line corresponds to minimum T_R, and is tangential to the curve at the speed for $(L/D)_{max}$.

Procedures for establishing minimum and maximum airspeeds with an available amount of power (P_A) are similar to those for the case when the available thrust (T_A) was supplied.

However even if power is independent of airspeed (prop aircraft) we now have to establish the intersection points graphically or numerically (since we can't make a quadratic equation that can be solved analytically).





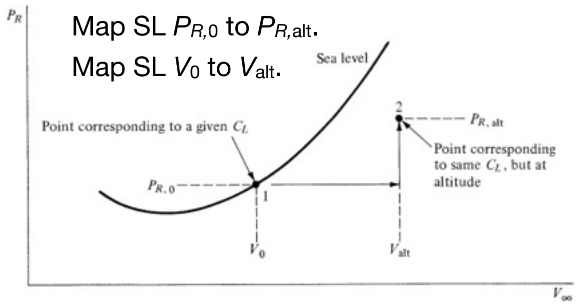
Anderson

Altitude effect on power required

Now since at SL, $P_{R,0} = \sqrt{\frac{2}{\rho_0}\frac{W^3}{S}\frac{C_D^2}{C_L^3}}$ and at altitude $P_{R,\text{alt}} = \sqrt{\frac{2}{\rho}\frac{W^3}{S}\frac{C_D^2}{C_L^3}}$ obtain $P_{R,\text{alt}} = P_{R,0}\sqrt{\frac{\rho_0}{\rho}} = \frac{P_{R,0}}{\sqrt{\sigma}}$

Likewise $V_0 \equiv V_e = \sqrt{\frac{2}{\rho_0}\frac{W}{S}\frac{1}{C_L}}$ $V_{\text{alt}} = \sqrt{\frac{2}{\rho}\frac{W}{S}\frac{1}{C_L}}$ $V_{\text{alt}} = V_e\sqrt{\frac{\rho_0}{\rho}} = \frac{V_e}{\sqrt{\sigma}}$

Map SL $P_{R,0}$ to $P_{R,\text{alt}}$.
Map SL V_0 to V_{alt}.



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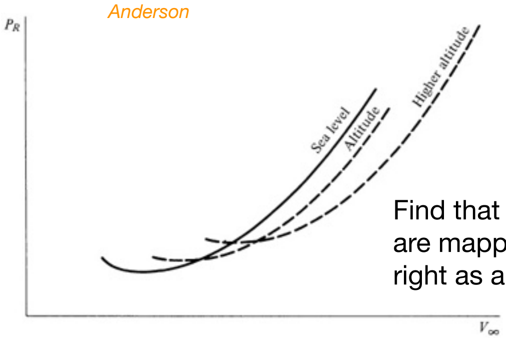
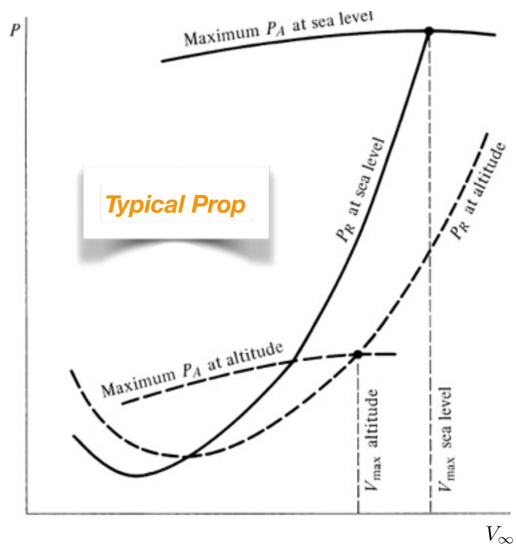
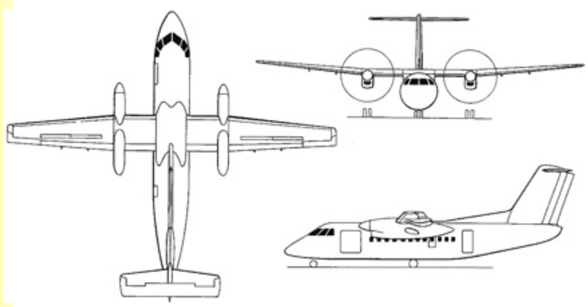


Find that P_R–V curves are mapped up and to the right as altitude increases.

Propeller-driven aircraft typically have their maximum speeds at SL:



Example – Bombardier Dash 8 regional turboprop



Supplied data:

W = 155 kN, S = 54.4 m², b = 25.9 m

Estimated $C_{D,0}$ = 0.02, aircraft efficiency factor e = 0.80

Engines 2x PW turboprop, SL static power P_0 = 1.53 GW each

Power model $\frac{P_A}{P_0} = \left(\frac{\rho}{\rho_{SL}}\right)^{0.7} = \sigma^{0.7}$

Estimate

1. Minimum power required for level flight at 5 km altitude
2. Airspeeds: TAS and EAS at this condition
3. Maximum SL airspeed

Aerodynamic parameters $A = \frac{b^2}{S} = \frac{25.9^2}{54.4} = 12.3$ $K = \frac{1}{\pi A e} = \frac{1}{\pi \times 12.3 \times 0.80} = 0.0323$

$$C_L^* = \sqrt{\frac{C_{D,0}}{K}} = \sqrt{\frac{0.02}{0.0323}} = 0.787 \qquad \left(\frac{C_L}{C_D}\right)^* = \frac{1}{\sqrt{4C_{D,0}K}} = \frac{1}{\sqrt{4 \times 0.02 \times 0.0323}} = 19.7$$

$$\frac{W}{S} = \frac{155}{54.4}\ \text{kPA} = 2850\ \text{Pa}$$

1. Minimum power required for level flight at 5km

$$\left(\frac{P}{W}\right)_{\min} = 1.755\left(\frac{2}{\rho}\frac{W}{S}\right)^{1/2}\left(C_{D,0}K^3\right)^{1/4} \qquad \rho = 0.7364\ \text{kg/m}^3 \qquad \sigma = \frac{\rho}{\rho_0} = \frac{0.7364}{1.225} = 0.6011$$

Example – Bombardier Dash 8 regional turboprop

$$\left(\frac{P}{W}\right)_{\min} = 1.755\left(\frac{2}{0.7364}\times 2850\right)^{1/2}\left(0.02\times 0.0323^3\right)^{1/4}\ \mathrm{W/N} = 4.42\ \mathrm{W/N}$$

$$P_{\min} = 155\times 4.42\ \mathrm{kW} = 686\ \mathrm{kW}$$

Check: available power $P_A = \sigma^{0.7}P_0 = 0.6011^{0.7}P_0 = 0.700\times 2\times 1530\ \mathrm{kW} = 2143\ \mathrm{kW}$ ✔ OK

2. TAS and EAS at this condition

At minimum power $V = u_{\text{min power}}V^* = 0.760V^* = 0.760\sqrt{\frac{2}{\rho}\frac{W}{S}\frac{1}{C_L^*}}$

$$= 0.760\times\sqrt{\frac{2}{0.7364}\times 2850\times\frac{1}{0.787}}\ \mathrm{m/s} = 75.4\ \mathrm{m/s}$$

Always $V_e = \sqrt{\sigma}V = \sqrt{0.6011}\times 75.4\ \mathrm{m/s} = 58.4\ \mathrm{m/s}$

3. Estimated maximum speed at SL

Maximum available power P_A = 2 × 1530 kW = 3060 kW.

Power required to fly is $P_R = W\sqrt{\frac{2}{\rho}\frac{W}{S}\frac{C_D^2}{C_L^3}}$

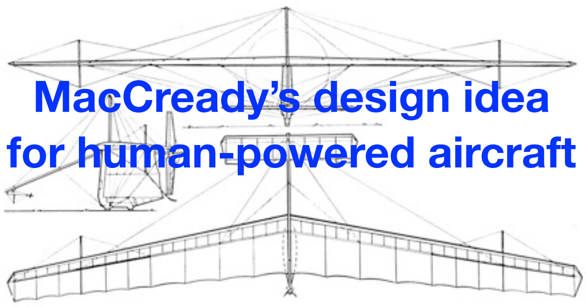
$C_D = C_{D,0} + KC_L^2$ and $C_L = \frac{2}{\rho}\frac{W}{S}\frac{1}{V^2}$

Solve P_A = P_R iteratively/graphically for V.

This occurs close to 163 m/s, accept V = 163 m/s.

V m/s	C_L	C_D	$\sqrt{C_D^2/C_L^3}$	P_R kW
100	0.4653	0.027	0.8505	899
200	0.1163	0.0204	0.5151	5446
150	0.2068	0.0214	0.2274	2404
160	0.1818	0.0211	0.2719	2875
163	0.1751	0.021	0.2864	3028

MacCready's design idea for human-powered aircraft



Paul MacCready got the basic idea for the first really successful HPA designs by considering hang-glider flight.

Hang-gliders are powered by gravity. If a pilot and glider have a mass of 90kg and sinking speed V_S = 1.2m/s, the power supplied by gravity is $P = mgV_S$ = 90 × 9.81 × 1.2 W = 1.06 kW.

To sustain level flight, a human powerplant would then need to sustain about 1 kW. But even elite cyclists can only maintain about 400 W over extended periods.

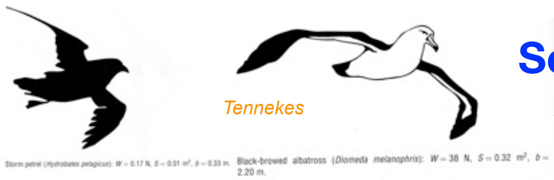
Since $P_R = \frac{W}{C_L/C_D}\sqrt{\frac{2}{\rho}\frac{W}{S}\frac{1}{C_L}}$ **MacCready's aim was to reduce wing loading *W/S* while keeping *W* fixed.**

MacCready reasoned that if the wing area S were increased by a factor of 9 over that for a hang-glider but W kept much the same then $(W/S)^{1/2}$ would fall by a factor of 3, as would the power requirement, P_R.

That would reduce the power required to 1060/3 W = 353 W, within the capacity of an elite cyclist.

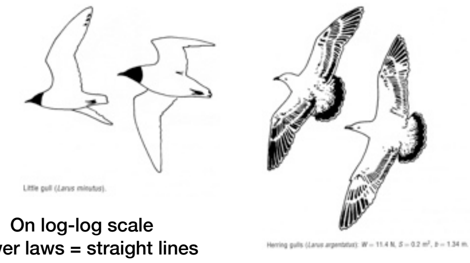
The reduction in W/S was achieved, and designs by the team went on to break all the records and win a number of major prizes.

The challenges presented and the efforts of the team to overcome them are well described in the book *Gossamer Odyssey* by Morton Grosser.



Scaling laws

a.k.a. Allometry



Consider the idea that all sea-birds are similar in structure.

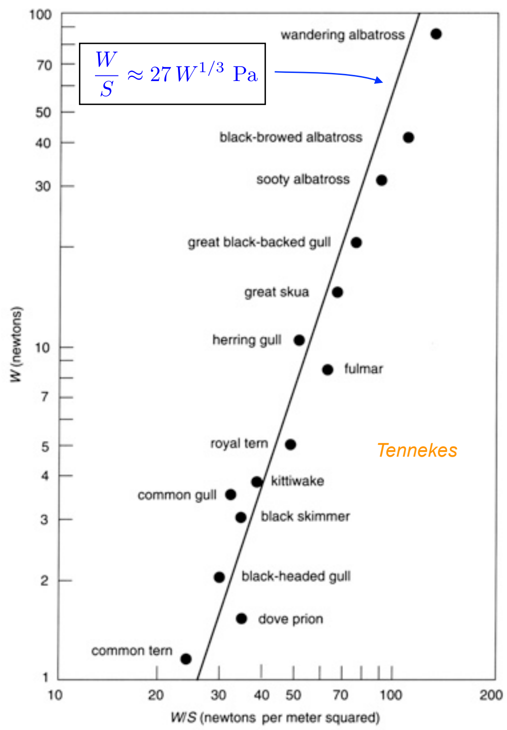
And that they have related cruise performance.

	W	S	W/S	V m/sec	V mph
Common tern	1.15	0.050	23	7.8	18
Dove prion	1.70	0.046	37	9.9	22
Black-headed gull	2.30	0.075	31	9.0	20
Black skimmer	3.00	0.089	34	9.4	21
Common gull	3.67	0.115	32	9.2	21
Kittiwake	3.90	0.101	39	10.1	23
Royal tern	4.70	0.108	44	10.7	24
Fulmar	8.20	0.124	66	13.2	30
Herring gull	9.40	0.181	52	11.7	26
Great skua	13.5	0.214	63	12.9	29
Great black-billed gull	19.2	0.272	71	13.6	31
Sooty albatross	28.0	0.340	82	14.7	33
Black-browed albatross	38.0	0.360	106	16.7	38
Wandering albatross	87.0	0.620	140	19.2	43

$S \propto b^2 \quad W \propto b^3 \longrightarrow \frac{W}{S} \propto b \longrightarrow \frac{W}{S} \propto W^{1/3}$

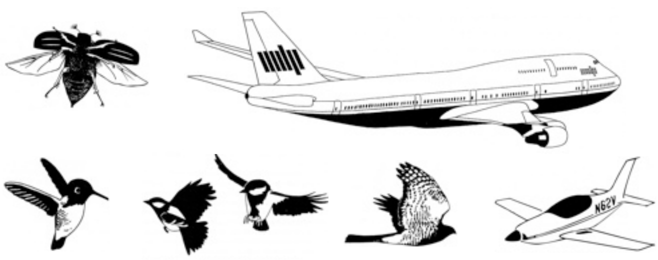
$b \propto W^{1/3}$

On log-log scale
power laws = straight lines



The Great Flight Diagram

Now consider the idea that cruising performance in all powered flight is related.



First we expand the range of weights (over an immense 11-decade range!) and change the constant of the previous power law curve, moving it to the right to better fit all the data.

Finally, think about flight speed. $L = W = \frac{1}{2}\rho V^2 S C_L$

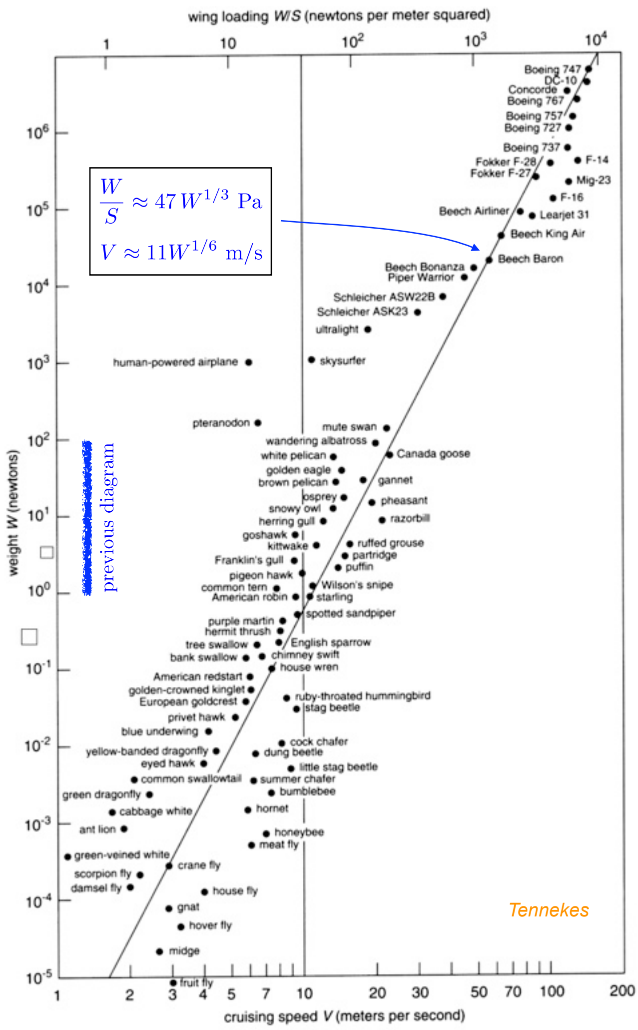
Representative values: $\rho \simeq 1.25\ \mathrm{kg/m^3} \quad C_{L_{\mathrm{cruise}}} \simeq 0.6$

$W \simeq 0.38 V^2 S \longrightarrow \frac{W}{S} \simeq 0.38 V^2$

used to obtain the lower scale on the diagram from *W/S*.

Recall $\frac{W}{S} \propto W^{1/3}$, hence now also $V \propto W^{1/6}$

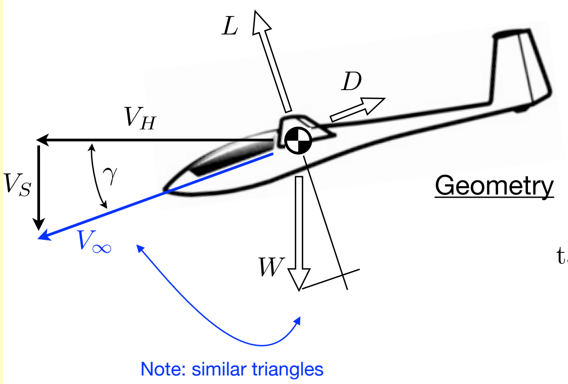
Considering the huge range of *W*, it is equally reasonable to use either of these power laws to correlate the data.



Gliding flight

Range and endurance in gliding flight

Gliding flight is a special case where there is no engine power supplied. Gravitational potential energy is dissipated to drag power. The vertical velocity component is the sink speed V_S.

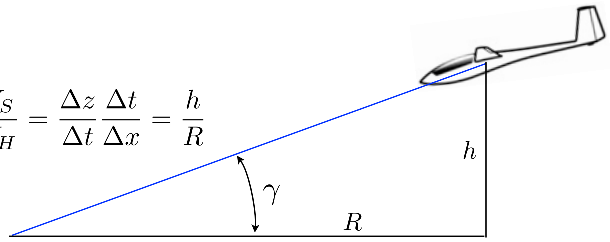


Force equilibrium

$$L = W\cos\gamma \qquad V_\infty = \sqrt{\frac{2}{\rho}\frac{W}{S}\cos\gamma\frac{1}{C_L}}$$
$$D = W\sin\gamma$$

Geometry

$$\tan\gamma = \frac{D}{L} = \frac{V_S}{V_H} = \frac{\Delta z}{\Delta t}\frac{\Delta t}{\Delta x} = \frac{h}{R}$$



Rate of energy dissipation = rate of loss of potential energy $DV_\infty = WV_\infty\sin\gamma = WV_S$

1. To maximize range R for a given height loss h (minimize γ), maximize L/D. **i.e. fly @ $(C_L/C_D)_{max}$**

$$L = W\cos\gamma = \tfrac{1}{2}\rho V_\infty^2 S C_L^* \qquad C_L^* = \left(\frac{C_{D,0}}{K}\right)^{1/2}$$

Now if (as is typical) $\cos\gamma \to 1$

$$V_\infty = \left(\frac{2}{\rho}\frac{W}{S}\frac{\cos\gamma}{C_L^*}\right)^{1/2} \qquad V_{\infty,\text{max range}} = V^* = \left[\frac{2}{\rho}\frac{W}{S}\left(\frac{K}{C_{D,0}}\right)^{1/2}\right]^{1/2}$$

Note that at fixed W/S, the TAS for best range increases with altitude but the glide angle γ and IAS do not. Alternatively, increasing W/S does not alter the best glide angle γ or distance travelled for given height loss.

Range and endurance in gliding flight

2. To maximize endurance for a given height loss (or, maximize rate of climb in rising air), minimize V_S.

$$V_S = V_\infty \sin\gamma = V_\infty \frac{D}{W} = V_\infty \tfrac{1}{2}\rho V_\infty^2 \frac{S}{W} C_D = \frac{\rho}{2}\frac{S}{W}\left(\frac{2}{\rho}\frac{W}{S}\cos\gamma\frac{1}{C_L}\right)^{3/2} C_D = \left(\frac{2}{\rho}\frac{W}{S}\right)^{1/2} (\cos\gamma)^{3/2}\frac{C_D}{C_L^{3/2}}$$

Typically, glide angle γ is small, so assume $\cos\gamma \to 1 \;\Rightarrow\; V_S \approx \left(\frac{2}{\rho}\frac{W}{S}\right)^{1/2}\frac{C_D}{C_L^{3/2}}$ **i.e. fly @ $(C_L^{3/2}/C_D)_{max}$**

For large glide angles, use $\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$

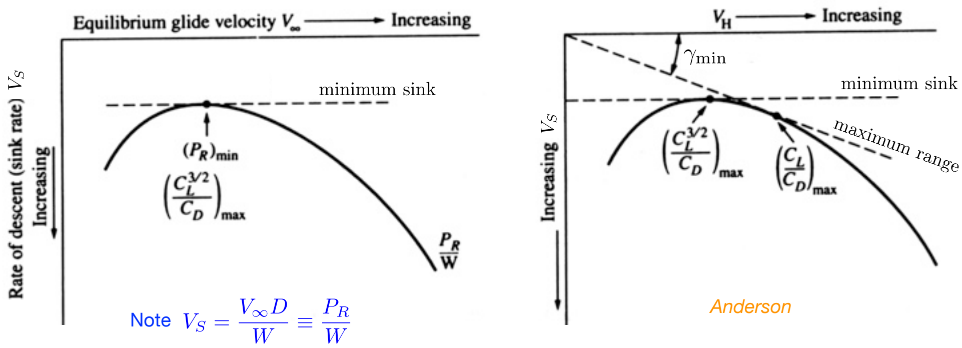
One can show that $C_{L,\text{min sink}} = \sqrt{\frac{3C_{D,0}}{K}}$ hence $V_{\infty,\text{min sink}} = \left[\frac{2}{\rho}\frac{W}{S}\left(\frac{K}{3C_{D,0}}\right)^{1/2}\right]^{1/2}$

Fly at what would be the minimum-power airspeed in level flight, i.e. 0.760 V^*.
Fly slower than for maximum range.

Recall: rate of energy dissipation = rate of loss of potential energy = gravity power.
We dissipate potential energy at the slowest rate by gliding at $(C_L^{3/2}/C_D)_{max}$.

Glider performance information is often plotted in the form of a 'glide hodograph' which is sink rate V_S plotted against either V_H or V_∞ (which are very similar).

This is just another way of showing the information contained in the drag polar.



Note $V_S = \frac{V_\infty D}{W} \equiv \frac{P_R}{W}$

Anderson

Glide testing to estimate drag polar information

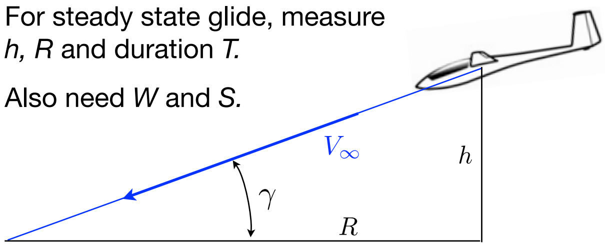
First set aircraft trim to achieve $(L/D)^*$, i.e. to achieve maximum range for a given height drop.

At $(L/D)^*$ we have $C_D = C_D^* = 2C_{D,0}$ $\qquad C_{D,0} = C_D^*/2$

and $C_L = C_L^* = \sqrt{\frac{C_{D,0}}{K}}$ $\qquad K = C_{D,0}/{C_L^*}^2$

For steady state glide, measure *h, R* and duration *T*.

Also need *W* and *S*.



$\gamma = \tan^{-1}(h/R)$ (typically $\gamma \to 0$ and $\cos\gamma \to 1$)

Sink speed $V_S = h/T$

Air speed $V_\infty = \sqrt{h^2+R^2}/T$

Energy dissipation $DV_\infty = WV_S$ or $D = \frac{WV_S}{V_\infty}$

$$D = \tfrac{1}{2}\rho V_\infty^2 S C_D = W\sin\gamma \qquad C_D = \frac{2}{\rho}\frac{D}{S}\frac{1}{V_\infty^2} = \frac{2}{\rho}\frac{W}{S}\frac{V_S}{V_\infty^3}$$

$$L = \tfrac{1}{2}\rho V_\infty^2 S C_L = W\cos\gamma \qquad C_L = \frac{2}{\rho}\frac{W}{S}\frac{\cos\gamma}{V_\infty^2}$$

These statements all hold true regardless of aircraft trim state (i.e. where on the drag polar we are flying).

After trimming to obtain best range, use the measurements to obtain C_D and C_L, i.e. C_D^* and C_L^*, hence first $C_{D,0}$ then K.

AIRCRAFT PERFORMANCE

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Range and endurance in powered flight

Range (distance flown) and endurance (time flown)

First, the simple principles:

Two idealized engine classes:

1. fuel mass flow rate $\dot{m}_f$ is proportional to power (typical of **prop**eller engine incl. turboprop);
2. fuel mass flow rate $\dot{m}_f$ is proportional to thrust (typical of **jet** engine).

To maximize endurance, minimize the weight of fuel consumed per unit time, i.e. $\dot{m}_f$

To maximize range, minimize the weight of fuel consumed per unit distance, i.e. $\dot{m}_f \Delta t/\Delta s = \dot{m}_f / V_\infty$

Recall:

For power-type propulsion we use power-specific fuel consumption, PSFC, given the symbol c_p.

SI units are kg/W.s. Note the values below are in mg/W.s

Typical PSFCs, c_p: lbm/bhp/hr {mg/W.s}	(Range) Cruise	(Endurance) Loiter
Piston-prop (fixed pitch)	0.4 {.068}	0.5 {.085}
Piston-prop (variable pitch)	0.4 {.068}	0.5 {.085}
Turboprop	0.5 {.085}	0.6 {.101}

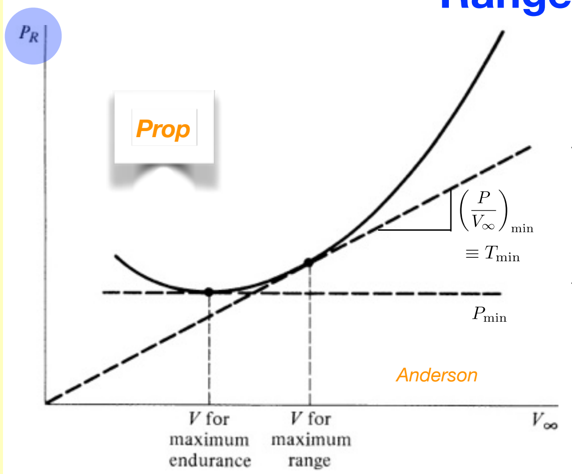
For thrust-type propulsion we use thrust-specific fuel consumption, TSFC, given the symbol c_t.

SI units are kg/N.s. Note the typical values below are in mg/N.s

Typical TSFCs, c_t: lbm/lbf/hr {mg/N.s}	Cruise	Loiter
Pure turbojet	0.9 {25.5}	0.8 {22.7}
Low-bypass turbofan	0.8 {22.7}	0.7 {19.8}
High-bypass turbofan	0.5 {14.1}	0.4 {11.3}

Raymer

Range and endurance



Propeller+piston, best characterized by power output

PSFC c_p = mass of fuel consumed per unit power per unit time.

<u>Maximum endurance: minimise kg of fuel per second</u>

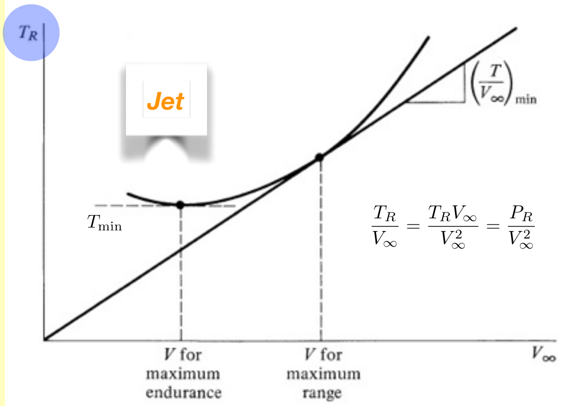
kg fuel/second ∝ $c_p \times P_S$. (P_S is shaft power)

Fly at minimum-power speed, i.e. $(C_L^{3/2}/C_D)_{max}$.

<u>Maximum range: minimize kg of fuel per m travelled</u>

kg fuel/m ∝ (kg fuel/sec)/(m/sec) = $c_p \times P_S / V_\infty = c_p \times T_R/\eta_{pr}$.

Fly at minimum-drag (and thrust) speed, V^*, $(C_L/C_D)_{max}$.



Jet, best characterized by thrust output

TSFC c_t = mass of fuel consumed per unit thrust per unit time.

<u>Maximum endurance: minimise kg of fuel per second</u>

kg fuel/second ∝ $c_t \times T_R$.

Fly at minimum-drag (and thrust) speed, V^*, $(C_L/C_D)_{max}$.

<u>Maximum range: minimize kg of fuel per m travelled</u>

kg fuel/m ∝ (kg fuel/sec)/(m/sec) = $c_t \times T_R / V_\infty$.

Now $V_\infty = \sqrt{\frac{2}{\rho}\frac{W}{S}\frac{1}{C_L}}$ so $\frac{T_R}{V_\infty} = \frac{W}{V_\infty (C_L/C_D)} \propto \frac{1}{C_L^{1/2}/C_D}$

Fly at minimum power/kinetic energy speed, i.e. $(C_L^{1/2}/C_D)_{max}$.

Three optimal aerodynamic ratios and flight speeds

The ratios $C_L^{3/2}/C_D \propto L/(DV)$, $C_L/C_D \propto L/D$ and $C_L^{1/2}/C_D \propto VL/D$ can all be considered as functions of dimensionless speed $u=V/V^*$ where V^* is the minimum-drag speed.

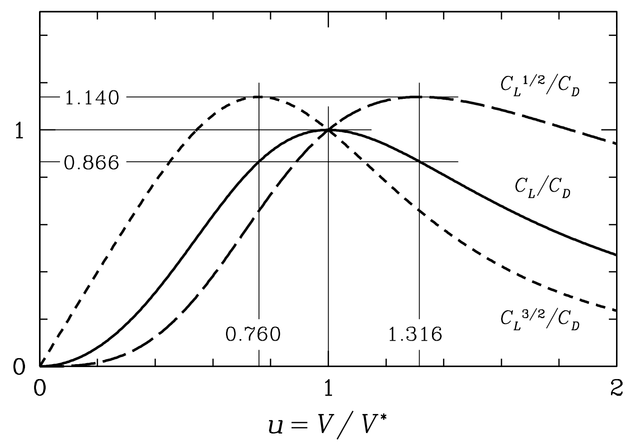
We already know

$$\frac{L}{D} = \left(\frac{C_L}{C_D}\right)^* \frac{2}{u^2+u^{-2}} = \left(\frac{C_L}{C_D}\right)^* \frac{2u^2}{u^4+1}$$

and $$\frac{L}{DV} \propto \frac{1}{u}\frac{2u^2}{u^4+1} = \frac{2u}{u^4+1}$$ Speed function associated with best prop endurance

so $$\frac{VL}{D} \propto u\frac{2u^2}{u^4+1} = \frac{2u^3}{u^4+1}$$ Speed function associated with best jet range

Plotting and then analysing these functions we can find various useful ratios associated with the maxima which are tabulated below.



Function	*Dimensionless*	*V/V*, at max*	*(L/D)/(L/D)*, at max*	*C_L/C_L^*, at max*
L/(DV)	$C_L^{3/2}/C_D$	$(1/3)^{1/4} = 0.760$	$(3/4)^{1/2} = 0.866$	$3^{1/2} = 1.732$
L/D	C_L/C_D	1	1	1
(VL)/D	$C_L^{1/2}/C_D$	$(3)^{1/4} = 1.316$	$(3/4)^{1/2} = 0.866$	$(1/3)^{1/2} = 0.577$

Recall: $C_L^* = (C_{D,0}/K)^{1/2}$; $(C_L/C_D)^* = 1/(4C_{D,0}K)^{1/2}$; $V^* = [(2/\rho)(W/S)(1/C_L^*)]^{1/2}$.

Range and endurance

Range; based on weight of fuel consumed per unit distance. Minimize $\dot{m}_{\mathrm{fuel}}\mathrm{d}t/\mathrm{d}x = \dot{m}_{\mathrm{fuel}}/V_\infty$

$$\dot{m}g\frac{\mathrm{d}t}{\mathrm{d}x} = -\frac{\dot{W}}{V} = -\frac{\mathrm{d}W}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = -\frac{\mathrm{d}W}{\mathrm{d}x} \equiv -\frac{\mathrm{d}W}{\mathrm{d}R}$$

where R is range. Hence $\frac{\mathrm{d}R}{\mathrm{d}W} = -\frac{V}{\dot{m}g}$

Jet $\dot{m}g = \mathrm{g}\, c_t\, T$ and $T = \frac{W}{L/D}$

$$\frac{\mathrm{d}R}{\mathrm{d}W} = -\frac{V}{\dot{m}g} = -\frac{V}{\mathrm{g}c_t\, T} = -\frac{1}{\mathrm{g}c_t}V\frac{L}{D}\frac{1}{W} \quad \text{i.e.} \quad R = -\int_{W_i}^{W_f}\frac{1}{\mathrm{g}c_t}V\frac{L}{D}\frac{\mathrm{d}W}{W} = \int_{W_f}^{W_i}\frac{1}{\mathrm{g}c_t}V\frac{L}{D}\frac{\mathrm{d}W}{W}$$

where W_i is the initial, W_f is the final aircraft weight for a flight segment.

Now assuming c_t, V, L/D are all constants: $R = \frac{1}{\mathrm{g}c_t}V\frac{L}{D}\ln\frac{W_i}{W_f}$

$=(9/16)^{1/4}$

We already know that VL/D is largest when we fly at $(C_L^{1/2}/C_D)_{\max}$., i.e. $L/D=(3/4)^{1/2}(L/D)^*$ and $V=3^{1/4}V^*$.

$$R_{\max} = \left(\frac{27}{16}\right)^{1/4}\frac{1}{\mathrm{g}c_t}V^*\left(\frac{L}{D}\right)^*\ln\frac{W_i}{W_f} = \frac{1.140}{\mathrm{g}c_t}V^*\left(\frac{L}{D}\right)^*\ln\frac{W_i}{W_f}$$

η_{pr} is propeller efficiency, $\eta_{\mathrm{pr}} \approx 0.8$.

Prop

$$\dot{m}g = \mathrm{g}c_p P_S = \frac{\mathrm{g}c_p DV}{\eta_{\mathrm{pr}}} = \frac{\mathrm{g}c_p TV}{\eta_{\mathrm{pr}}} = \frac{\mathrm{g}c_p WV}{\eta_{\mathrm{pr}}(L/D)} \qquad \frac{\mathrm{d}R}{\mathrm{d}W} = -\frac{V}{\dot{m}g} = -\frac{\eta_{\mathrm{pr}}(L/D)}{\mathrm{g}c_p W}$$

$$\mathrm{d}R = -\frac{\eta_{\mathrm{pr}}}{\mathrm{g}c_p}\frac{L}{D}\frac{\mathrm{d}W}{W}$$

Hence, to maximise, assuming L/D and c_p const: $R_{\max} = \frac{\eta_{\mathrm{pr}}}{\mathrm{g}c_p}\left(\frac{L}{D}\right)^*\ln\frac{W_i}{W_f}$

(As originally derived by French engineer Breguet – the generic label for all these related equations.)

Range and endurance

Endurance; based on weight of fuel consumed per unit time. Minimize $\dot{m}_{\mathrm{fuel}}$

$$\dot{m}g = -\dot{W} = -\frac{\mathrm{d}W}{\mathrm{d}t} \equiv -\frac{\mathrm{d}W}{\mathrm{d}E}$$

where E is endurance. Hence $\frac{\mathrm{d}E}{\mathrm{d}W} = -\frac{1}{\dot{m}g}$

Jet $\dot{m}g = \mathrm{g}\, c_t\, T$ Then, working similarly to previously for jet range, we have

$\mathrm{d}E = -\frac{1}{\mathrm{g}c_t}\frac{L}{D}\frac{\mathrm{d}W}{W}$ Assuming c_t and L/D const: $E = \frac{1}{\mathrm{g}c_t}\frac{L}{D}\ln\frac{W_i}{W_f}$ $E_{\max} = \frac{1}{\mathrm{g}c_t}\left(\frac{L}{D}\right)^*\ln\frac{W_i}{W_f}$

Prop As above: $\dot{m}g = \mathrm{g}c_p P_S = \frac{\mathrm{g}c_p DV}{\eta_{\mathrm{pr}}} = \frac{\mathrm{g}c_p TV}{\eta_{\mathrm{pr}}} = \frac{\mathrm{g}c_p WV}{\eta_{\mathrm{pr}}(L/D)}$

$\mathrm{d}E = -\frac{\eta_{\mathrm{pr}}}{\mathrm{g}c_p}\frac{1}{V}\frac{L}{D}\frac{\mathrm{d}W}{W}$ $E = \frac{\eta_{\mathrm{pr}}}{\mathrm{g}c_p}\frac{1}{V}\frac{L}{D}\ln\frac{W_i}{W_f}$ which we know is maximized when flying at $(C_L^{3/2}/C_D)_{\max}$,

where $L/D = 0.866(L/D)^*$, $V = 0.760V^*$: $E_{\max} = \left(\frac{27}{16}\right)^{1/4}\frac{\eta_{\mathrm{pr}}}{\mathrm{g}c_p}\frac{1}{V^*}\left(\frac{L}{D}\right)^*\ln\frac{W_i}{W_f}$

To maximize endurance for propeller aircraft, typically want to have small induced drag (hence high aspect ratio), and fly at low altitude. This **Breguet** Atlantique ASW aircraft is one example.

NB: In a design problem, we will often have the range or endurance specified and need to find the weight fraction W_i/W_f and from this $W_{\mathrm{fuel}} = W_i\,(1-W_f/W_i)$.

Range and endurance: the three aspects of design

It is interesting to note that the range and endurance equations engage the three central aircraft design disciplines of (1) aerodynamics (2) propulsion (3) structures.

E.g. the jet range equation (all range/endurance equations are of similar form):

Propulsion design

$$R = V \times \frac{L}{D} \times \frac{1}{gc_t} \times \ln \frac{W_i}{W_f}$$

Aerodynamic design

Structural design

The goal of each design discipline is to maximise its ratio in this equation.

But in reality all the disciplines are coupled.

AIRCRAFT PERFORMANCE

Range and endurance

Summary

Optimal conditions for range and endurance all demand flying at fixed points on the drag polar.

Range, *R*

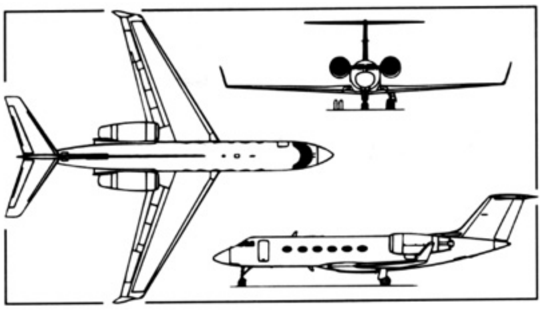
Type	Equation	Optimal flight strategy
Jet	$R = \frac{1}{gc_t} V \frac{L}{D} \ln \frac{W_i}{W_f}$	Maximized at high altitude (high $V \Rightarrow$ low ρ). Fly at 1.316V^*, 0.577C_L^*, 0.866$(C_L/C_D)^*$.
Prop	$R = \frac{\eta_{\text{pr}}}{gc_p} \frac{L}{D} \ln \frac{W_i}{W_f}$	Independent of altitude. Fly at V^*, C_L^*, $(C_L/C_D)^*$.

A conceptual difficulty is that for jet aircraft, the above suggests that range increases indefinitely as altitude is increased (reducing ρ, increasing V^*). Eventually, Mach number and/or propulsion system limits start to take effect and invalidate this simple model. But for now, it's good enough.

Endurance, *E*

Type	Equation	Optimal flight strategy
Jet	$E = \frac{1}{gc_t} \frac{L}{D} \ln \frac{W_i}{W_f}$	Independent of altitude. Fly at V^*, C_L^*, $(C_L/C_D)^*$.
Prop	$E = \frac{\eta_{\text{pr}}}{gc_p} \frac{1}{V} \frac{L}{D} \ln \frac{W_i}{W_f}$	Maximized at low altitude (low $V \Rightarrow$ high ρ). Fly at 0.760V^*, 1.732C_L^*, 0.866$(C_L/C_D)^*$.

Example – Grumman Gulfstream IV business jet



W_0 = 324 kN, S = 88.3 m^2, b = 23.7 m.

TSFC for cruise: c_t = 18 mg/N.s = 18×10^{-6} kg/N.s.

<u>If the aircraft's maximum takeoff weight is 40% fuel, estimate the maximum cruise range at altitude h = 11km.</u>

Wing loading at MTOW, W_0/S = 324/88.3 kPa = 3.67 kPa.

Previously derived C_L^* = 0.505 and $(C_L/C_D)^*$ = 16.8.

Know $R_{\max} = \dfrac{1.140}{gc_t} V^* \left(\dfrac{L}{D}\right)^* \ln\dfrac{W_i}{W_f}$ and $\dfrac{W_i}{W_f} = \dfrac{1}{0.6}$ Tables say ρ = 0.3640 kg/m^3 @ 11km.

Now, V^* is a function of W/S. It is appropriate to use an average value of $(W/S)_{\text{avg}}$ = 0.8 × 3.67 kPa = 2.94 kPa.

$$V^*_{\text{avg}} = \left(\frac{2}{\rho}\left(\frac{W}{S}\right)_{\text{avg}}\frac{1}{C_L^*}\right)^{1/2} = \left(\frac{2}{0.3640}\times 2940 \times \frac{1}{0.505}\right)^{1/2} \text{m/s} = 178.8\,\text{m/s}$$

We note $V(C_L^{1/2}/C_D)_{\max}$ = 1.316 V^* = 235.4 m/s. This is a Mach number of 235.4/295.2 @ 11 km, M = 0.80.

$$R = \frac{1.140}{9.81\times 18\times 10^{-6}} \times 178.8 \times 16.8 \times \ln\frac{1}{0.6}\,\text{m} = 9.906\times 10^6\,\text{m} = 9906\,\text{km} \equiv 5347\,\text{nm}$$

Manufacturer's specification gives R = 7815 km.

We have not allowed for takeoff, climb, landing, and reserves. Certainly we are in the right ballpark.

Range and endurance as areas on a graph

Recall the differential equations we developed for endurance and range.

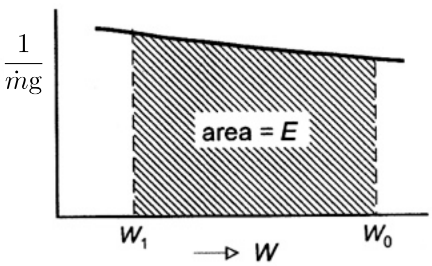
<u>**Endurance**; based on weight of fuel consumed per unit time.</u>

$$\dot{m}g = -\dot{W} = -\frac{\mathrm{d}W}{\mathrm{d}t} \equiv -\frac{\mathrm{d}W}{\mathrm{d}E}$$ where E is endurance. Hence $$\frac{\mathrm{d}E}{\mathrm{d}W} = -\frac{1}{\dot{m}g}$$

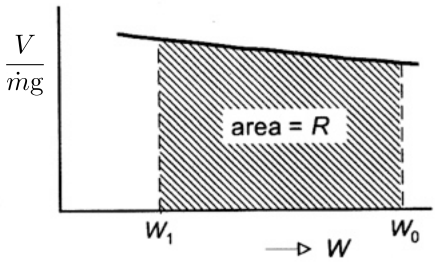
<u>**Range**; based on weight of fuel consumed per unit distance.</u>

$$\dot{m}g\frac{\mathrm{d}t}{\mathrm{d}x} = -\frac{\dot{W}}{V} = -\frac{\mathrm{d}W}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = -\frac{\mathrm{d}W}{\mathrm{d}x} \equiv -\frac{\mathrm{d}W}{\mathrm{d}R}$$ where R is range. Hence $$\frac{\mathrm{d}R}{\mathrm{d}W} = -\frac{V}{\dot{m}g}$$

We made various idealisations so that we could develop simple first-pass estimates (the Breguet equations), but in general – and probably more accurately – the equations can be seen as the basis for integral estimates (either graphical or numerical). For this approach the idealisations are not needed.



(a) Endurance

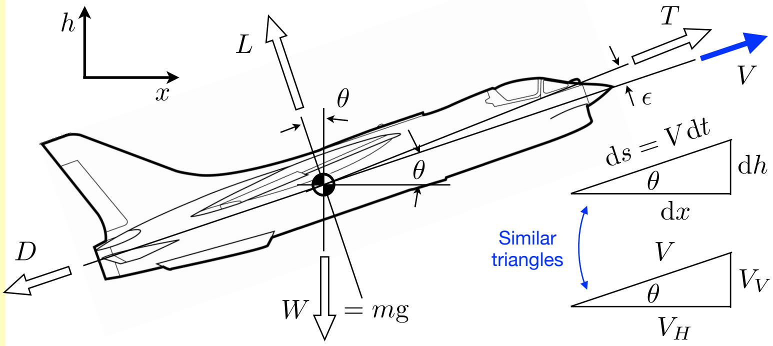


(b) Range

Torenbeek & Wittenberg

Steady climbing flight

Steady climbing flight



Returning to

$$\frac{\dot{V}}{g} = \frac{T\cos\epsilon - D}{W} - \sin\theta$$

$$\frac{\dot{\theta}}{g} = \frac{T\sin\epsilon + L}{VW} - \frac{\cos\theta}{V}$$

with $\dot{V} = 0, \quad \dot{\theta} = 0$ and $\epsilon = 0$

The central difference to the steady level flight case is that now $\theta \neq 0$.

The two equations of equilibrium reduce to $\dfrac{T-D}{W} = \sin\theta$ and $L = W\cos\theta$

From the first equation there are two related parameters

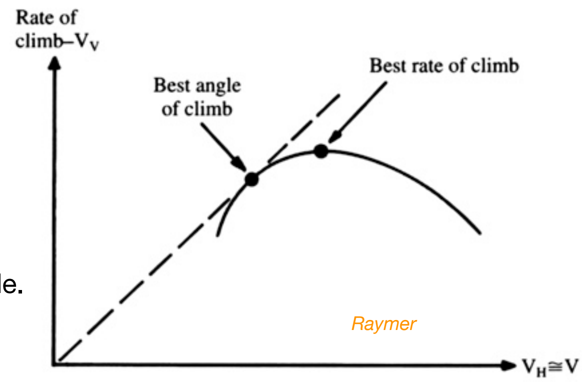
Angle of climb $\quad \sin\theta = \dfrac{V_V}{V} = \dfrac{T-D}{W} = \dfrac{P-DV}{WV}$

Rate of climb $\quad V_V = \dfrac{dh}{dt} = \dfrac{(T-D)V}{W} = \dfrac{P-DV}{W}$

Their maxima occur at different airspeeds.

$\dfrac{(T-D)}{W}$ called the specific excess thrust, gives climb angle.

$\dfrac{(T-D)V}{W}$ called the specific excess power, is climb speed.



Climb rate and climb angle

Angle of climb $\sin\theta = \frac{V_V}{V} = \frac{T-D}{W} = \frac{P-DV}{WV} = \frac{T}{W} - \frac{D}{W} \approx \frac{T}{W} - \frac{D}{L} = \frac{T}{W} - \frac{1}{C_L/C_D}$

Rate of climb $V_V = \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{(T-D)V}{W} = \frac{P-DV}{W}$

The **greatest angle of climb** will occur for a flight speed where the specific excess thrust is largest.

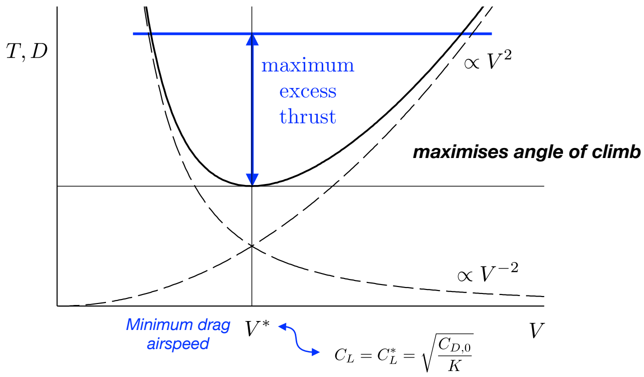
For a jet-propelled aircraft where the thrust is almost independent of speed, this occurs when we fly *near the minimum drag speed*, i.e. at $(C_L/C_D)_{max} = (C_L/C_D)^*$. Result becomes exact if we assume $\cos\theta \to 1$.

The **greatest rate of climb** will occur for a flight speed where the specific excess power is largest.

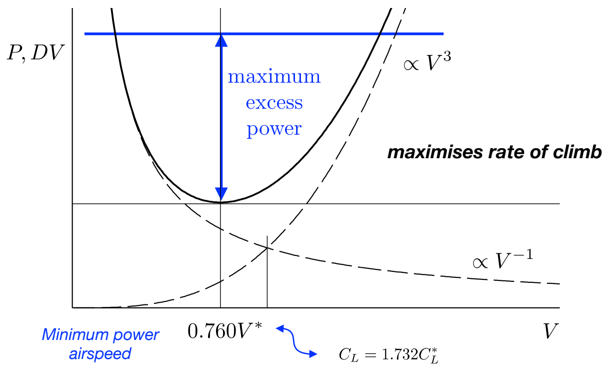
For a piston/prop aircraft where the power is almost independent of speed, this occurs when we fly *near the minimum power speed*, i.e. at $(C_L^{3/2}/C_D)_{max}$. Again, result becomes exact if we assume $\cos\theta \to 1$.

These are the two easiest cases to analyse. Also possible to analytically solve for maximum climb rate for a jet. Problem of maximum climb angle for a piston+prop must be solved graphically or numerically.

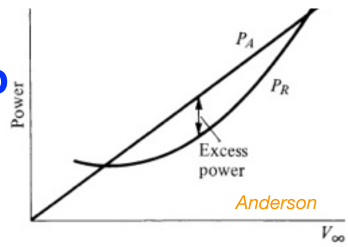
Jet: thrust approx. independent of speed.



Piston+prop: power approx. independent of speed.



Jet aircraft maximum rate of climb



★

$$V_V = \frac{V_\infty(T-D)}{W} = V_\infty\left[\frac{T}{W} - \frac{D}{W}\right] = V_\infty\left[\frac{T}{W} - \tfrac{1}{2}\rho V_\infty^2\left(\frac{W}{S}\right)^{-1} C_{D,0} - \frac{W}{S}\frac{2K}{\rho V_\infty^2}\right]$$

$$\frac{\mathrm{d}V_V}{\mathrm{d}V_\infty} = \frac{T}{W} - \frac{3}{2}\rho V_\infty^2\left(\frac{W}{S}\right)^{-1} C_{D,0} + \frac{W}{S}\frac{2K}{\rho V_\infty^2} = 0 \qquad V_\infty^2 - \frac{2(T/W)(W/S)}{3\rho C_{D,0}} - \frac{4K(W/S)^2}{3\rho^2 C_{D,0} V_\infty^2} = 0$$

Recall $(L/D)_{\max} = 1/\sqrt{4KC_{D,0}}$, multiply through by V_∞^2.

$$V_\infty^4 - \frac{2(T/W)(W/S)}{3\rho C_{D,0}} V_\infty^2 - \frac{(W/S)^2}{3\rho^2 C_{D,0}^2 (L/D)_{\max}^2} = 0 \qquad \text{Let} \quad Q \equiv \frac{W/S}{3\rho C_{D,0}}, \quad x \equiv V_\infty^2.$$

$$x^2 - 2\frac{T}{W}Qx - \frac{3Q^2}{(L/D)_{\max}^2} = 0 \qquad x = \frac{2(T/W)Q \pm \sqrt{4(T/W)^2Q^2 + 12Q^2/(L/D)_{\max}^2}}{2}$$

$$= \frac{T}{W}Q\left(1 \pm \sqrt{1 + \frac{3}{(L/D)_{\max}^2 (T/W)^2}}\right)$$

Only '+' gives a real result.

$$V_{(V_V)_{\max}} = \left(\frac{(T/W)(W/S)}{3\rho C_{D,0}}\left[1 + \sqrt{1 + \frac{3}{(L/D)_{\max}^2 (T/W)^2}}\right]\right)^{1/2} \qquad \text{Let} \quad Z \equiv 1 + \sqrt{1 + \frac{3}{(L/D)_{\max}^2 (T/W)^2}}.$$

$$V_{(V_V)_{\max}} = \left(\frac{(T/W)(W/S)Z}{3\rho C_{D,0}}\right)^{1/2}$$

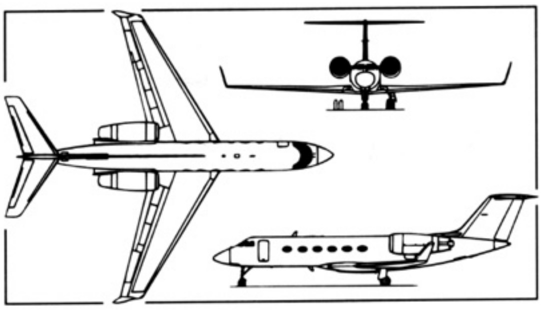
This is a flight speed, not rate of climb itself.

Now go back and substitute this into ★

Eventually

$$(V_V)_{\max} = \left[\frac{(W/S)Z}{3\rho C_{D,0}}\right]^{1/2}\left(\frac{T}{W}\right)^{3/2}\left[1 - \frac{Z}{6} - \frac{3}{2(T/W)^2(L/D)_{\max}^2 Z}\right]$$

Example – Grumman Gulfstream IV business jet



W_0 = 324 kN, S = 88.3 m², b = 23.7 m, W_0/S = 3.67 kPa.

Engines 2x R–R turbofan, SL static thrust T_0 = 65 kN each.

Thrust model $\frac{T_A}{T_0} = 0.5\left(\frac{\rho}{\rho_{SL}}\right)^{0.7} = 0.5\sigma^{0.7}$ $\quad C_{D,0}$=0.015

Previously derived C_L* = 0.505 and (C_L/C_D)* = 16.8.

Estimate maximum angle and rate of climb at SL, T_A = 65kN.

1. Max climb angle at SL

$$T_{R,\min} = \frac{W}{(C_L/C_D)^*} = 19.3\,\text{kN} \qquad \sin\theta = \frac{T_A - T_{R,\min}}{W} = \frac{65-19.3}{324} = 0.141 \qquad \theta = 8.1^\circ$$

2. Max climb rate at SL

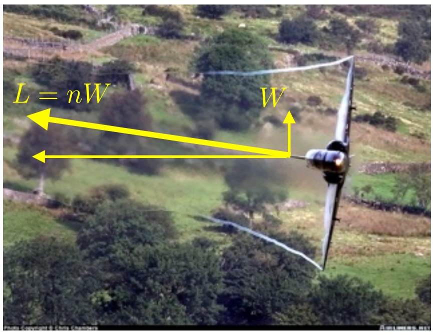
$$\frac{T_A}{W} = \frac{65}{324} = 0.201 \qquad Z = 1+\sqrt{1+\frac{3}{(L/D)^2_{\max}(T/W)^2}} = 1+\sqrt{1+\frac{3}{16.8^2\times 0.201^2}} = 2.214$$

$$(V_V)_{\max} = \left[\frac{(W/S)Z}{3\rho C_{D,0}}\right]^{1/2}\left(\frac{T}{W}\right)^{3/2}\left[1-\frac{Z}{6}-\frac{3}{2(T/W)^2(L/D)^2_{\max}Z}\right]$$

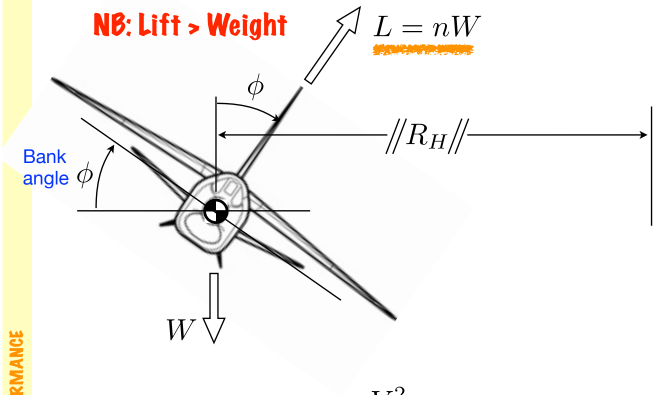
$$= \left[\frac{3670\times 2.124}{3\times 1.225\times 0.015}\right]^{1/2}\times 0.201^{3/2}\times\left[1-\frac{2.124}{6}-\frac{3}{2\times 0.201^2\times 16.8^2\times 2.124}\right]\text{m/s} \quad = 19.8\,\text{m/s}$$

Anderson chapter 6

Turning flight



Steady coordinated turn



In turning manoeuvres, we seek to change the heading of the aircraft, and a commonly-used basis for analysis is to assume that the aircraft executes a coordinated horizontal banked turn of constant radius R_H and at constant speed V. A component of lift has to balance the centrifugal force to allow this.

For now we will also assume that $\epsilon = 0$, and hence that the thrust makes no direct contribution to turning. It is simple enough to restore this term if required.

Note the use of the load factor n to describe the lift. This corresponds to the increased wing lift (and bending load).

Horizontal $\quad L \sin\phi = m\dfrac{V^2}{R_H} = mV\omega$

Vertical $\quad L\cos\phi = nW\cos\phi = W, \quad n\cos\phi = 1, \quad \cos\phi = \dfrac{1}{n}, \quad \phi = \cos^{-1}\left(\dfrac{1}{n}\right)$

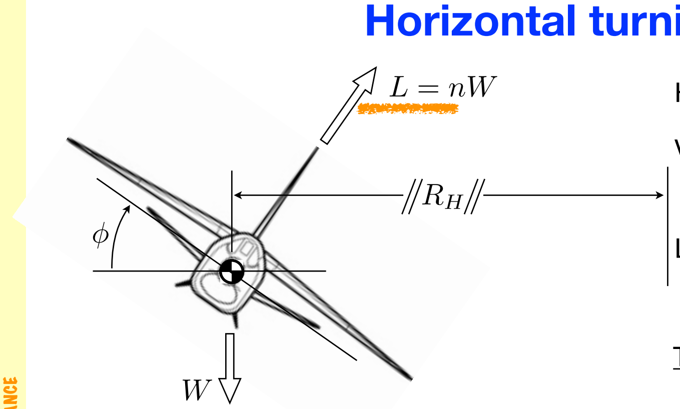
$$\cos^2\phi + \sin^2\phi = 1, \quad \sin^2\phi + \frac{1}{n^2} = 1, \quad \sin^2\phi = 1 - \frac{1}{n^2} = \frac{n^2-1}{n^2}, \quad \sin\phi = \frac{\sqrt{n^2-1}}{n}$$

Substitute into horizontal component equation:

$$\frac{nW\sqrt{n^2-1}}{n} = mV\omega, \quad mg\sqrt{n^2-1} = mV\omega, \quad \omega = \frac{g\sqrt{n^2-1}}{V}$$

Equivalently: $$R_H = \frac{V^2}{g\sqrt{n^2-1}}$$

Horizontal turning performance



Horizontal equilibrium $\quad L\sin\phi = m\dfrac{V^2}{R_H} = mV\omega$

Vertical equilibrium $\quad L\cos\phi = nW\cos\phi = W$

Leading to $\quad n = \dfrac{1}{\cos\phi} \qquad \sin\phi = \dfrac{\sqrt{n^2-1}}{n}$

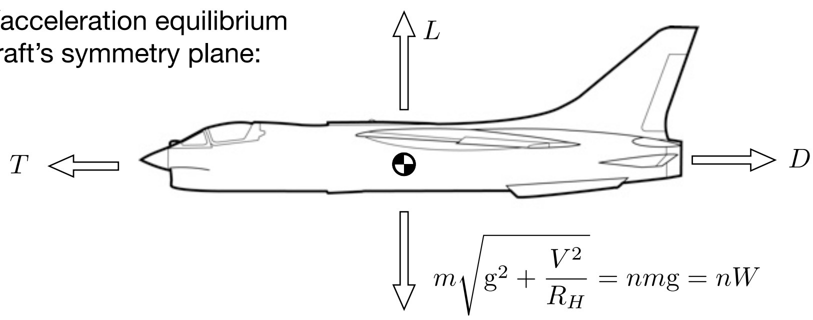
The lift required to turn is greater than in level flight.

We obtained the following for rate and radius of turn: $\quad \omega = \dfrac{g\sqrt{n^2-1}}{V} \qquad R_H = \dfrac{V^2}{g\sqrt{n^2-1}}$

Typically we wish to either maximise the turn rate or minimise the turn radius. The first usually more important.

To consider the thrust requirement, we use force equilibrium in the tangential direction, which is $T = D$, the same as in steady level flight. However, we no longer have $L = W$, but $L = nW$, instead.

Force/acceleration equilibrium in aircraft's symmetry plane:



Now, $L = nW = \frac{1}{2}\rho V^2 S C_L$

or $\quad C_L = \dfrac{2}{\rho}\dfrac{W}{S}\dfrac{n}{V^2}$

Horizontal turning performance

Because the turn is coordinated (and assumed to be of comparatively large radius) we can continue to use the drag polar relationship already developed: $C_D = C_{D,0} + KC_L^2$

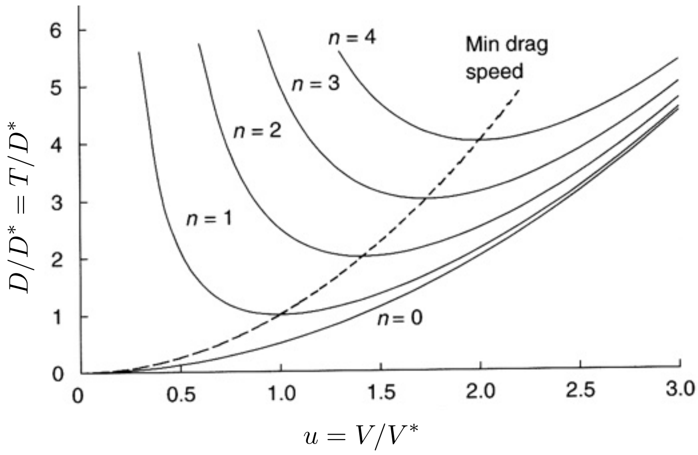
$$T = D = \frac{nW}{C_L/C_D} = nW\left(\frac{C_{D,0}}{C_L} + KC_L\right) \quad \text{i.e.} \quad \frac{T}{W} = n\left(\frac{C_{D,0}}{C_L} + KC_L\right)$$

(same as steady level flight but with *nW* replacing *W*.)

Working similarly to steady level flight: $\left(\frac{T}{W}\right)_{\min} = \frac{n}{(C_L/C_D)^*} = n\sqrt{4C_{D,0}K}$

$V_{\text{min drag}} = \sqrt{n}\,V^*$ where *V** is for steady level flight.

Turning makes *n* > 1, resulting in more induced drag at a given speed and hence requiring more thrust.



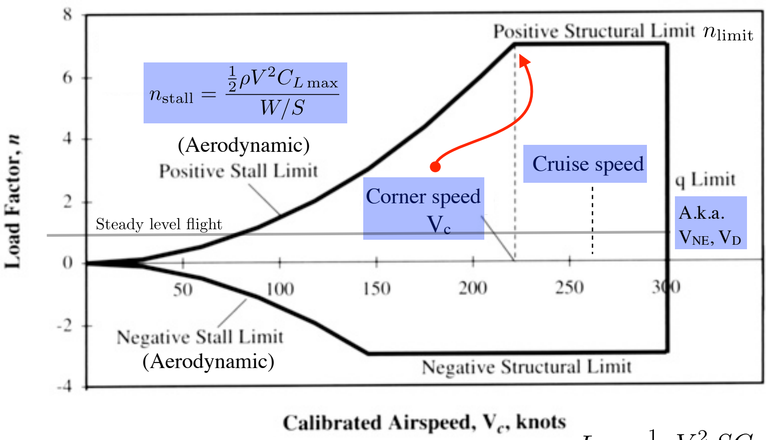
Eshelby

<u>Also, turning increases the structural load on the aircraft.</u>

The *V—n* diagram

The load factor *n* is derived from *L* = *nW*, i.e. *n* = *L/W,* and describes how much load the aircraft structure carries compared to the case in level flight (which has *n* = 1).

The <u>*V–n* diagram expresses the speed/load-factor envelope</u> of the aircraft as determined by aerodynamic constraints (e.g. stall) and structural strength.



NB: another (colloquial) name for load factor is "g-force".

Typical structural limit load factors.

Aircraft Type	Load Factor
General Aviation (normal)	$-1.25 \le n \le 3.1$
General Aviation (utility)	$-1.8 \le n \le 4.4$
General Aviation (acrobatic)	$-3.0 \le n \le 6.0$
Homebuilt	$-2 \le n \le 5$
Commercial Transport	$-1.5 \le n \le 3.5$
Fighter	$-4.5 \le n \le 7.75$

<u>At low speeds</u>, $n_{\max}$ is a function of $C_{L\max}$: $n = \frac{L}{W} = \frac{\frac{1}{2}\rho V_\infty^2 S C_L}{W}$ i.e. a stall limit: $n_{\max} = \frac{1}{2}\rho V_\infty^2 \frac{C_{L,\max}}{W/S}$

So at low speeds, there is a maximum magnitude of *n* that the aircraft can attain before it <u>stalls</u>.

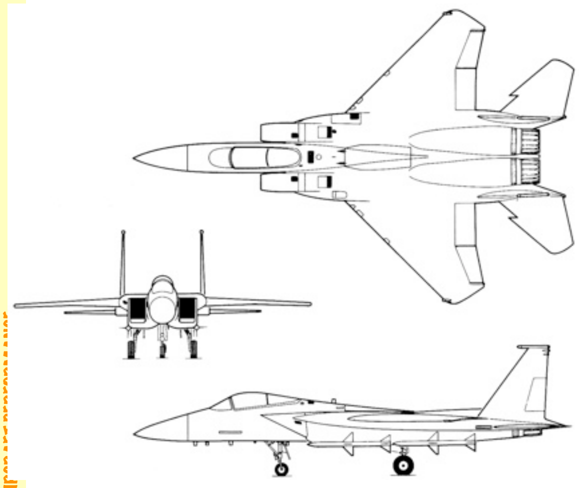
<u>At higher speeds</u>, there is a maximum magnitude of *n* that the aircraft can attain before it <u>fails structurally.</u>

Finally there may be an <u>upper speed limit</u> (or *q* limit) derived either from shock wave effects, structural flutter (a structural dynamic/aerodynamic interaction), or by consideration of transient loads that could be imposed by sudden control inputs.

At <u>corner speed</u>, aerodynamic and structural limits are simultaneously met. $V_c = \sqrt{\frac{2}{\rho}\frac{W}{S}\frac{n_{\text{limit}}}{C_{L\max}}}$

(This gives the maximum turn rate.)

Example



Fighter aircraft with wing loading *W/S* = 5.2 kPa and mass 30 t.

Wing aspect ratio *A* = 3.5.

(Subsonic) airplane efficiency factor *e* = 0.9.

Zero-lift drag coefficient $C_{D,0}$ = 0.015.

$$K = 1/(\pi A e) = 0.1011$$

$$(C_L/C_D)^* = 1/\sqrt{4 C_{D0} K} = 12.84$$

$$W = mg = 294.3 \text{ kN} \qquad S = W/(W/S) = 56.60 \text{ m}^2$$

<u>What is minimum amount of thrust required to fly level?</u>

$$T_{\text{min, level flight}} = \frac{W}{(C_L/C_D)^*}$$

***T*$_{min}$ = 22.9 kN.**

$$C_L^* = \sqrt{C_{D,0}/K} = 0.3583$$

<u>What is coefficient of lift required for a sustained level turn of *n* = 5, *M* = 0.8, at altitude *h* = 9km?</u>

From tables

$$\sigma = 0.3813 \qquad \rho = \sigma\rho_0 = 0.4671 \text{ kg/m}^3$$

$$a/a_0 = 0.8929 \qquad a = 303.9 \text{ m/s} \qquad V = Ma = 243.1 \text{ m/s}$$

$$L = nW \qquad C_L = \frac{2}{\rho}\frac{nW}{S}\frac{1}{V^2} = 1.884$$

***C*$_L$ = 1.88.**

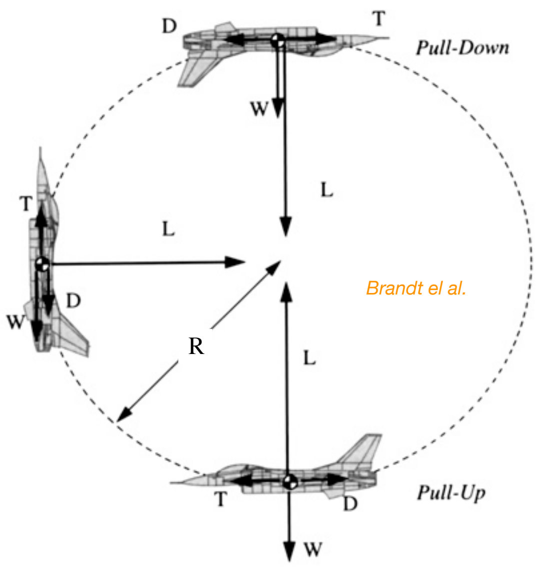
<u>What is the amount of thrust required for a sustained level turn of *n* = 5, *M* = 0.8, at altitude *h* = 9km?</u>

$$C_D = C_{D,0} + K C_L^2 = 0.3737 \qquad T = D = \tfrac{1}{2}\rho V^2 S C_D = 291.9 \text{ kN}$$

***T* = 292 kN.**

Turning in the vertical plane

1. Two central examples are pull-up and pull-down from level flight.



2. At Pull-up $m\frac{V_\infty^2}{R} = L - W$

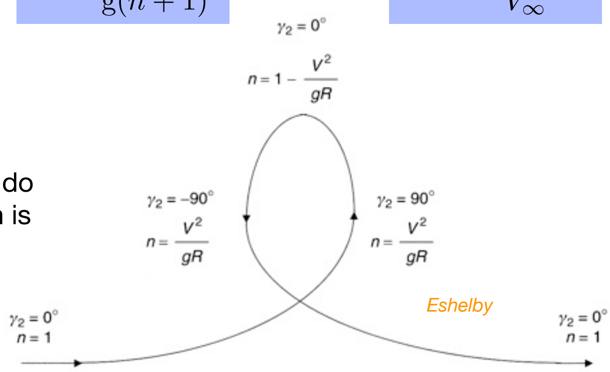
$$R = \frac{mV_\infty^2}{L-W} = \frac{W}{g}\frac{V_\infty^2}{L-W} = \frac{V_\infty^2}{g(L/W-1)} = \frac{V_\infty^2}{g(n-1)}$$

$$\omega = \frac{V_\infty}{R} = \frac{g(n-1)}{V_\infty}$$

3. At Pull-down $m\frac{V_\infty^2}{R} = L + W$

$$R = \frac{V_\infty^2}{g(n+1)} \qquad \omega = \frac{g(n+1)}{V_\infty}$$

4. One result is that looping manoeuvres typically do not have a true circular fight path, unless action is taken to ensure it.



Limiting cases for large load factor

1. Radius *R*

Level turn	Pull up	Pull down	Large-*n* limit (n≫1)
$R = \dfrac{V_\infty^2}{g\sqrt{n^2-1}}$	$R = \dfrac{V_\infty^2}{g(n-1)}$	$R = \dfrac{V_\infty^2}{g(n+1)}$	$R = \dfrac{V_\infty^2}{gn}$

2. Turn rate ω

Level turn	Pull up	Pull down	Large-*n* limit (n≫1)
$\omega = \dfrac{g\sqrt{n^2-1}}{V_\infty}$	$\omega = \dfrac{g(n-1)}{V_\infty}$	$\omega = \dfrac{g(n+1)}{V_\infty}$	$\omega = \dfrac{gn}{V_\infty}$

3. Dependence on aircraft and flight parameters

$$L = \tfrac{1}{2}\rho V_\infty^2 S C_L, \quad V_\infty^2 = \frac{2L}{\rho S C_L}$$

$$R = \frac{V_\infty^2}{gn} = \frac{2L}{\rho S C_L g n} = \frac{2L}{\rho S C_L g(L/W)} = \frac{2}{\rho C_L g}\frac{W}{S}$$

$$R_{\text{min}} = \frac{2}{\rho g (C_L)_{\text{max}}}\frac{W}{S}$$

$$\omega = \frac{gn}{V_\infty} = \frac{gn}{\sqrt{2L/(\rho S C_L)}} = \frac{gn}{\sqrt{[2n/(\rho C_L)](W/S)}} = g\sqrt{\frac{n\rho C_L}{2(W/S)}}$$

$$\omega_{\text{max}} = g\sqrt{\frac{\rho (C_L)_{\text{max}} n_{\text{max}}}{2(W/S)}}$$

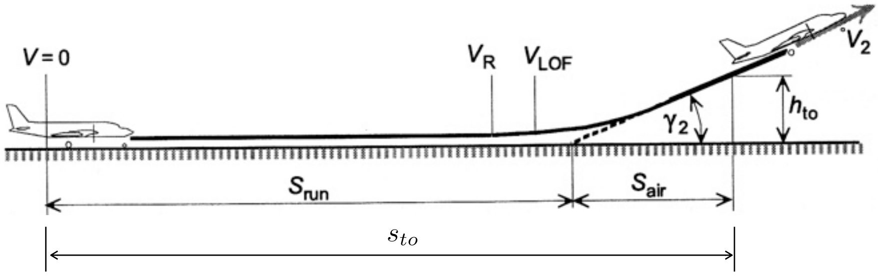
Conclusion: best turn performance is obtained by aircraft with (a) low wing loading *W/S*; (b) large maximum lift coefficient $C_{L\max}$; (c) greatest structural strength $n_{\max}$; and – for sustained turns (d) large thrust loading *T/W.*

Takeoff and landing performance

AIRCRAFT PERFORMANCE

Normal take-off

A normal takeoff is made with all engines operating.



Torenbeek & Wittenberg

1. During the ground run the aircraft rolls from standstill with nosewheel on the ground so that the angle of attack is nearly constant (and typically a value that makes aerodynamic drag small).
2. Rotation is initiated at speed V_R. When $L = W$, the aircraft becomes airborne at speed V_{LOF}.
3. During the initial airborne phase $L > W$ and there is acceleration normal to the flight path, but soon afterwards the angle of climb settles to a constant value and the undercarriage is retracted to reduce drag. The aircraft accelerates to "safety speed" V_2 which is at least k_{to} (usually a factor of 1.2) times the stall speed.
4. Takeoff is said to be completed after the aircraft reaches a "screen height" h_{to}, which is 35 ft for commercial aircraft and 50 ft for military aircraft.
5. The total runway length must exceed the ground roll plus the distance required to clear the screen.

$$s_{\text{to}} = s_{\text{run}} + s_{\text{air}}$$

$$V_2 > k_{\text{to}} V_{\text{stall}} = 1.2 \sqrt{\frac{2}{\rho} \frac{W}{S} \frac{1}{C_{L,\max}}} \qquad C_{L_2} \leq C_{L,\max}/1.44$$

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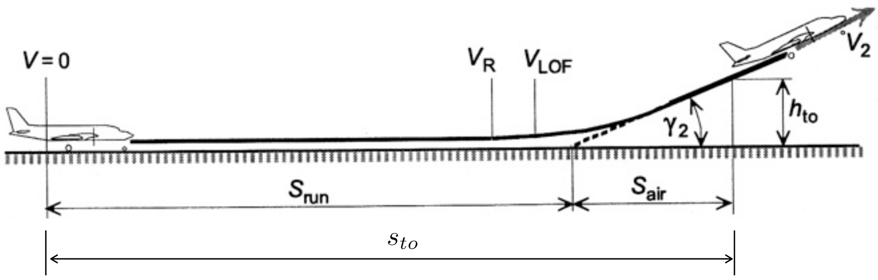
AIRCRAFT PERFORMANCE

Normal take-off

Simplified analysis assumes $V_{LOF} = V_2$ and that a steady climb with speed V_2 and angle γ_2 starts at lift-off.



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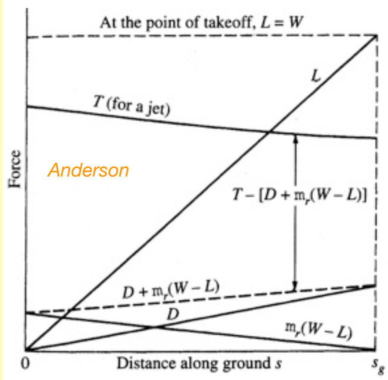
$$s_{\text{to}} \approx \frac{V_2^2}{2\bar{a}} + \frac{h_{\text{to}}}{\tan \gamma_2}$$

where $\bar{a}$ is the average acceleration over the ground.



A more complete analysis has to account for a drop in engine thrust with speed, the friction of the tyres on the ground, and the fact that the vertical reaction force (hence friction force) falls as the aircraft starts to produce lift even while on the ground.

For now we take these complexities into account by applying a simple reduction factor on thrust and say that

$$\bar{a} = r_T \frac{T_{\text{to}}}{m} \qquad \text{or} \qquad \bar{a} = r_T \frac{T_{\text{to}}}{W} g$$

where for a jet aircraft $r_T \approx 0.8 - 0.9$ but is lower again for propeller aircraft where thrust falls with speed.

Normal take-off

$$s_{\text{to}} \approx \underbrace{\frac{V_2^2}{2\bar{a}}}_{s_{run}} + \underbrace{\frac{h_{\text{to}}}{\tan\gamma_2}}_{s_{air}} \qquad \bar{a} = r_T \frac{T_{\text{to}}}{W} g$$

Recall for steady climb $\sin\gamma_2 = \frac{T-D}{W} = \frac{T}{W} - \frac{D}{W}$ where $W \to L = \frac{1}{2}\rho V_2^2 S C_{L_2}$ or $V_2^2 = \frac{2}{\rho}\frac{W}{S}\frac{1}{C_{L_2}}$

$$\sin\gamma_2 = \left(\frac{T}{W} - \frac{C_D}{C_L}\right)_2 \qquad C_{L_2} \approx C_{L,\max}/k_{\text{to}}^2 \quad \text{and} \quad C_D = C_{D,0} + K C_{L_2}^2$$

and $\sin\gamma_2 \to \tan\gamma_2$

finally

$$s_{\text{to}} \approx \underbrace{\frac{1}{\rho g r_T}\frac{1}{C_{L_2}}\frac{W}{S}\frac{W}{T_{\text{to}}}}_{s_{run}} + \underbrace{\frac{h_{\text{to}}}{T_2/W - (C_D/C_L)_2}}_{s_{air}}$$

Typically an additional 15% safety margin is added to this value (or any better estimate).

Note that T_2 may be significantly smaller than T_{to}, especially so for a propeller aircraft.

1. Ground run s_{run} increases quadratically with weight W and is reduced by either decreasing the wing loading W/S or increasing the thrust/weight ratio T_{to}/W, or both. Increasing weight also increases the air distance s_{air}.
2. Air density+temperature may have a significant effect on both ρ and T. High, hot take-offs are worst.
3. Increasing flap deflections will increase $C_{L\max}$ and hence C_{L2}, which reduces the first term but increases C_D/C_L and hence the second term. There is an optimum flap deflection which is typically less than the value used at landing, and hence $C_{L,\max,\text{to}} < C_{L,\max,\text{land}}$.

Engine failure during take-off

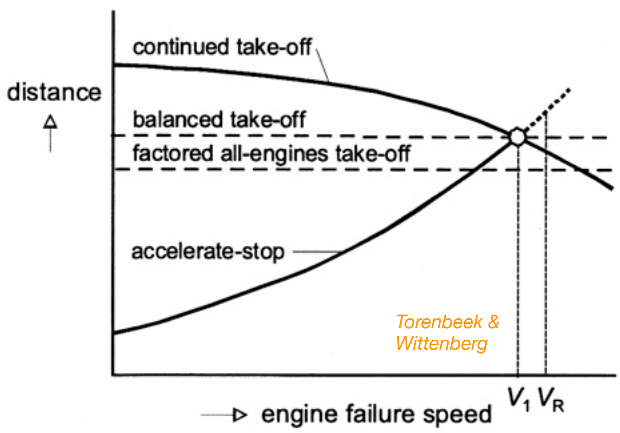
Engine failure (fortunately rare) will obviously lead to an aborted takeoff in a single-engine aircraft.

Multi-engined aircraft are designed so that they will still be able to climb with ***one engine inoperative*** (**OEI**).

If an engine fails then obviously total thrust is reduced but also the unpowered engine may contribute significant drag. There is typically a lateral asymmetry of thrust (and associated yawing moment) and the available control surface authority (as well as the pilot!) must be able to cope with this.

If engine failure is recognised at a sufficiently low speed (on the ground) then all engines are throttled back and all available braking (excluding thrust reversal) is used to de-accelerate the aircraft to a halt. The associated total distance on the runway is called the ***accelerate-stop distance***.

After a certain speed is reached it takes less runway distance to continue the takeoff and climb to screen height h_{to} than it does to brake to a halt. This speed, less than the rotation speed V_R, is called the ***decision speed*** V_1. Above V_1 the pilot will continue to take-off regardless of the runway length available.

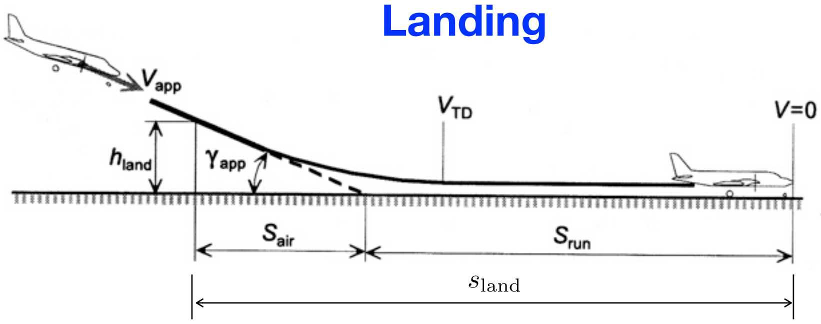


For any given engine failure speed, the total distance required required to accelerate to it and stop can be found, as can the distance required to continue and climb over screen height. The first increases with speed while the second decreases. The distance at which they are equal is called the ***balanced field length*** (**BFL**) and the associated speed is V_1.

The required runway length is the minimum of the BFL and 1.15 times the value estimated for the all-engines-operative case.

Because the aircraft's climbing capacity is reduced, there is typically also a regulated minimum climb gradient at V_2 with OEI. This is 2.4% for two engines, 2.7% for 3 engines and 3.0% for four engines.

Landing



Torenbeek & Wittenberg

1. During *landing approach* the aircraft flies at a steady speed $V_{app} > k_{app} V_{stall} = 1.3 V_{stall}$. The gradient γ_{app} is typically around 2.5° to 3° for commercial aircraft.
2. Once the *runway threshold height* h_{land} (typically 50 ft or 15 m) is reached the engines are throttled back and the pilot executes a *landing flare* or *round-out* to touch-down at V_{td}, typically 1.15 V_{stall}.
3. After touch-down of all the undercarriage elements the aircraft is slowed by wheel brakes (and perhaps airbrakes) until it comes to rest. While engine thrust reversal may be applied this is typically not included in an analysis designed to compute the minimum runway length required.

For a simplified analysis, $s_{\text{land}} = s_{\text{air}} + s_{\text{run}} \approx \dfrac{h_{\text{land}}}{\tan\gamma_{\text{app}}} + \dfrac{V_{\text{app}}^2}{2|\bar{a}|}$ where $V_{\text{app}} = 1.3\sqrt{\dfrac{2}{\rho}\dfrac{W}{S}\dfrac{1}{C_{L,\max}}}$

leading to

$$s_{\text{land}} \approx \frac{h_{\text{land}}}{\tan\gamma_{\text{app}}} + 1.69\frac{W/S}{\rho|\bar{a}|C_{L,\max}}$$

Note that the wing loading *W/S* may be much less than the maximum takeoff value, owing to fuel use.

Typical value of de-acceleration possible on a dry concrete runway is $|\bar{a}|/\mathrm{g} = 0.3 \text{ to } 0.5$.

Note also that, unlike the case for take-off, the thrust loading *T/W* does not come into account here.